Robust stability of polytopic time-inhomogeneous Markov jump linear systems [★]

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Abstract

The transition probabilities of jumps between operational modes of discrete-time Markov(ian) jump linear systems (dtMJLSs) are generally considered to be time-invariant, certain, and often completely known in the majority of dedicated studies. Still, in most real cases the transition probability matrix (TPM) cannot be computed exactly and is time-varying. In this article, we take into account the uncertainty and time-variance of the jump parameters by considering the underlying Markov chain as polytopic and time-inhomogeneous, i.e., its TPM is varying over time with variations that are arbitrary within a polytopic set of stochastic matrices. We show that the conditions used for time-homogeneous dtMJLSs are not enough to ensure the stability of the time-inhomogeneous system, and that perturbations on values of the TPM can make a stable system unstable. We present necessary and sufficient conditions for mean square stability (MSS) of polytopic time-inhomogeneous dtMJLSs, prove that deciding MSS on such systems is NP-hard and that MSS is equivalent to exponential MSS and to stochastic stability. We also derive necessary and sufficient conditions for robust MSS of dtMJLSs affected by polytopic uncertainties on transition probabilities and bounded disturbances.

Key words: Time-inhomogeneous Markov chains; Markov models; stochastic jump processes; robust stability

1 Introduction

To date, quite a few fundamental control issues, such as stability and stabilization, estimation and filtering, fault detection and diagnosis, have been addressed in the literature on discrete-time Markov(ian) jump linear systems (dtMJLSs), see Costa, Fragoso & Marques (2005), Zhang, Yang, Shi & Zhu (2016) as textbooks with important results and detailed examination of the general state of the art. However, as a crucial factor governing the behaviours of dtMJLSs, the transition probabilities (TPs) are generally considered to be time-invariant, certain, and often completely known in the majority of studies. Still, in most real cases the transition probability matrices (TPMs) are affected by global uncertainty due to random and systematic errors of measurement and numerical computation procedures (used to obtain the values of TPMs), by incomplete knowledge of some TPs (when adequate samples of the transitions are costly or time-consuming to obtain), and by abrupt and unpredictable time-variance (due to environmental factors, like for instance the wind perturbing the model of airspeed variation in a vertical take-off landing helicopter system, see Long & Yang (2013)). The polytopic time-inhomogeneous (PTI) model of the TPMs will allow us to take into account all the aforementioned aspects. Our choice of the PTI model is motivated by the fact that uncertainty and time-variance are intrinsic to the real-world systems, and all measurement and numerical analysis procedures give us confidence levels (determined

by accuracy and precision of the measuring instrument and/or numerical algorithm), which bound the possible values each TP can assume.

There are several alternatives to PTI model of TPs in the literature on uncertain dtMJLSs, but most of them typically account for either incomplete knowledge of TPMs, or time-variance. Specifically, the incomplete knowledge of stationary TPs can be represented or as norm-bounded (Karan, Shi & Kaya, 2006) or as polytopic uncertainties (Costa, Assumpção, Boukas & Marques (1999), where the precise values are not obtained and only the bounds of TPs are available), or as partially unknown TPs (Zhang, Boukas & Lam (2008), Zhang, Yang, Shi & Zhu (2016), where not all values are available). See Zacchia Lun (2017) for a comparison of these models of time-invariant uncertainties in TPs and for an overview of the related results on stability (De Souza, 2006; Zhang, Yang, Shi & Zhu, 2016). The uncertainties in time-inhomogeneous characteristics of TPs instead, in general, can be determined by either non-deterministic or stochastic variations. PTI dtMJLSs studied here, and dtMJLSs governed by piecewise homogeneous Markov chains subject to an arbitrary high-level switching signal (e.g., the signal with average dwell time approaching zero, see Sun, Zhao & Hill (2006), Colaneri (2009), Bolzern, Colaneri & De Nicolao (2010) and references therein) account for the first type of variations, while semi-Markov jump linear systems (semi-MJLSs, see Zhang, Leng & Colaneri (2016)), and piecewise homogeneous dtMJLSs with TPs themselves governed by a higher-level Markov chain (MC, see Zhang, Yang, Shi & Zhu (2016, Part II)) provide a rationale for stochastic variations. For dtMJLSs with TPMs taking values in the finite set and switching governed by possibly a priori unknown sequence, Lutz & Stilwell (2016) have presented necessary and sufficient conditions for uniform exponentially mean square stability and uniform stochastic disturbance attenuation, expressed as a set of finite-dimensional linear matrix inequalities (LMIs, see e.g. Boyd, El Ghaoui, Feron & Balakrishnan (1994) for a general discussion). It used time-varying quadratic Lyapunov function arguments. When the sojourn time in dtMJLSs' operational modes does not follow geometric distribution, the TPs are time-varying and have a "memory" property, resulting in so-called semi-MJLSs. See Zhang, Yang, Shi & Zhu (2016) for a formal introduction of such systems, and detailed treatment of stability and stabilization via semi-Markov kernel (where the probability density function of sojourn-time is dependent on both current and next system mode) and time-varying Lyapunov function approach. Noticeably, both semi-MJLSs and dtMJLSs with piecewise-constant TPs subject to a higher-level TPM require a prior knowledge of time-varying behaviour of transitions between operational modes of the system, in order to describe the involved stochastic variations. Furthermore, similarly to the case of piecewise homogeneous dtMJLSs governed by an arbitrary average dwell time

signal, the considered variations need to be in a finite set. This requirement implies the fundamental assumption that the TPs can be computed exactly. The PTI dtMJLSs model does not have such limitations, and has been already seen in works that have considered the robust stability problem. Notably, in Aberkane (2011) a sufficient condition for stochastic stability (SS) in terms of LMI feasibility problem was provided. The approach of Aberkane (2011) made use of a parameter dependent stochastic Lyapunov function. In Chitraganti, Aberkane & Aubrun (2013), instead, a sufficient condition for MSS of a dtMJLS with interval TPM, which in turn can be represented as a convex polytope (Hartfiel, 1998), was presented in relation to spectral radius. In general, before our contribution, described below, only sufficient stability conditions have been derived for dtMJLSs with time-inhomogeneous MCs having TPM arbitrarily varying within a polytopic set of stochastic matrices.

As a main contribution of this paper, we derive necessary and sufficient conditions for (robust) MSS of autonomous dtMJLSs governed by PTI MCs in both autonomous and affected by a bounded process noise settings. Such conditions require to decide whether the joint spectral radius (JSR) of a finite family of matrices is smaller than 1. While it is well known that the stability analysis problem for general switching systems (i.e. deciding whether the JSR is smaller than 1) is NP-hard (Tsitsiklis & Blondel, 1997), we prove that it is NP-hard even for the matrices structure deriving from our particular model. We also show that MSS is equivalent to exponential MSS (EMSS) and to SS. A preliminary and reduced version of the work on noiseless dtMJLSs with polytopic time-varying TPMs has been presented at the 55th IEEE Conference on Decision and Control (Zacchia Lun, D'Innocenzo & Di Benedetto, 2016), while a preliminary version of work on PTI dtMJLS with bounded process noise has been presented at the 20th IFAC World Congress (Zacchia Lun, D'Innocenzo & Di Benedetto, 2017). With respect to the preliminary versions, this paper provides a uniform treatment of the problem, with additional technical details and a revised version of some proofs, in order to simplify the presentation and improve the formal rigor. Notably, this paper adds Lemma 17, and revises the proofs of Proposition 9 and Theorems 11, 14 and 16. Furthermore, it provides a detailed analysis of robust stability for a realistic system of considerable size, with an emphasis on the gained knowledge and the computational effort.

The remainder of the paper is structured as follows. In Section 2 we introduce the notation used throughout this article and present the conceptual preliminaries necessary to a formal treatment of the topic. Then, in Section 3 we provide a mathematical model of dtMJLSs with PTI TPMs and different stability definitions relevant to our work. Next, Section 4 is devoted to the formal introduction of the notion of the joint spectral radius and its notable properties, used in the following Sections 5

and 6 to present the main technical results of this article on robust stability of PTI dtMJLSs with and without the process noise. Finally, Section 7 provides a practical numerical example and Section 8 concludes this work.

2   Notation and conceptual preliminaries

We denote by $\mathbb{F}$ the set of either real or complex numbers, and the sets of all nonnegative and all positive numbers are indicated by subscripts $_0$ and $_+$, respectively. We will be dealing with finite-dimensional linear spaces, for which all norms, denoted by $\|\cdot\|$, are equivalent (Kubrusly, 2001, Theorem 4.27), so the completeness of a normed linear space is preserved when the given norm is replaced by an equivalent one. We will use the variants of p-norms (Meyer, 2000, p. 274) (a.k.a. $\mathbb{L}^p$-norms), which $\forall \mathrm{x} \in \mathbb{F}^n$, and $\forall p \geq 1$, are defined as $\|\mathrm{x}\|_p \triangleq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. We denote by $\mathbb{F}^{m,n}$ a set of matrices with m rows, n columns, and entries in $\mathbb{F}$, or, equivalently, a set of linear maps between two linear spaces $\mathbb{F}^n$ and $\mathbb{F}^m$. We indicate by $\mathbb{F}_0^{n,n}$ a set of all positive semi-definite matrices of order n with entries in $\mathbb{F}$, and by $_\mathrm{N}\mathbb{F}_0^{n,n}$ the set of all N-sequences of square matrices in $\mathbb{F}_0^{n,n}$. The identity matrix of size n is denoted by I_n, while the zero matrix of the same size is denoted by 0_n. The operation of transposition is indicated by superscript ^{T}, the complex conjugation by overbar, while the conjugate transpose of a (complex) matrix is denoted by superscript *. For any square matrix A, its trace is denoted by $\mathrm{tr}(A)$, and the spectral radius by $\rho(A)$. To write concisely specific rows, columns and submatrices of a given matrix, we denote by $A_{i\bullet}$ the i-th row of a matrix $A = [a_{ij}]$, by $A_{[i,i+d]\bullet}$ the submatrix of A containing $d+1$ consecutive rows, starting from $A_{i\bullet}$, and by $A_{\bullet j}$ the j-th column of a matrix A. By its definition (Meyer, 2000, p. 280), any matrix norm satisfies subadditive property expressed by the triangle inequality stated $\forall A, B \in \mathbb{F}^{m,n}$ as $\|A+B\| \leq \|A\| + \|B\|$. As specific matrix norms, we will use the entry-wise norms ℓ_1 and ℓ_2 (Horn & Johnson, 2012, p. 341), denoted by $\|\cdot\|_{\mathbf{1}}$, $\|\cdot\|_{\mathbf{2}}$, and defined respectively as $\|A\|_{\mathbf{1}} \triangleq \|\mathrm{vec}(A)\|_1$, and $\|A\|_{\mathbf{2}} \triangleq \|\mathrm{vec}(A)\|_2 = \sqrt{\mathrm{tr}(A^*A)}$. We indicate by N the number of operational modes (a.k.a. discrete states) of the system, by $\mathbb{M}$ the related set of operational modes, and will extensively use a linear space made up of all N-sequences of either real or complex $m \times n$ matrices, denoted by $_\mathrm{N}\mathbb{F}^{m,n}$. The direct sum operation, denoted by $\oplus$, will be applied to sequences of square matrices, e.g. $\boldsymbol{A} = (A_i)_{i=1}^{N}$, to produce a block diagonal matrix, having the elements of $\boldsymbol{A}$ on the main diagonal blocks. The Kronecker product, denoted by $\otimes$, will be used together with transformation converting a matrix into a column vector, known as vectorization, denoted by $\mathrm{vec}(\cdot)$. Then, we indicate by $\mathrm{vec}^2(\cdot)$ a linear operator defined $\forall \boldsymbol{A} = (A_i)_{i=1}^{N} \in {}_\mathrm{N}\mathbb{F}^{m,n}$ as $\mathrm{vec}^2(\boldsymbol{A}) \triangleq [\mathrm{vec}(A_1), \mathrm{vec}(A_2), \ldots, \mathrm{vec}(A_N)]^T$. For the

space $_\mathrm{N}\mathbb{F}^{m,n}$, we define the following equivalent norms: $\|\boldsymbol{A}\|_1 \triangleq \sum_{i=1}^{N} \|A_i\|$, then $\|\boldsymbol{A}\|_2 \triangleq \sqrt{\sum_{i=1}^{N} \mathrm{tr}(A_i^* A_i)}$, and $\|\boldsymbol{A}\|_{\max} \triangleq \max_{i \in \mathbb{M}} \{\|A_i\|\}$. The linear spaces $_\mathrm{N}\mathbb{F}^{m,n}$ equipped with any of the above norms are uniformly homeomorphic to a finite-dimensional Banach space $\mathbb{F}^{Nmn}$ through the mapping $\mathrm{vec}^2(\cdot)$ (Costa et al., 2005, p. 17). Thus, all these normed linear spaces are complete.

The stochastic basis of dtMJLSs considered in this article is defined by the quadruple $(\Omega, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$, where Ω is the sample space, $\mathcal{G}$ is a corresponding σ-algebra of events, $(\mathcal{G}_k)$ is the filtration, and Pr is the probability measure. The jump variable is defined as $\theta_k : \Omega \to \mathbb{M}$, s.t. $\forall \omega \in \Omega$, with $\omega \triangleq \{(\phi_k, \chi_k) : k \in \mathbb{Z}_0, \phi_k \in \mathbb{M}, \chi_k \in \mathbb{F}^{n_\times}\}$, $\theta_k(\omega) = \phi_k$. The values $i \in \mathbb{M}$ of the jump variable θ_k are all measurable elementary events on $\mathcal{G}$. As a consequence, the indicator function $\mathbf{1}_{\{\theta_k=i\}}$ is s.t., for any $\omega \in \Omega$, one has $\mathbf{1}_{\{\theta_k=i\}}(\omega) = 1$ if $\theta_k(\omega) = i$, and 0 otherwise. The TP between the operational modes i, j in $\mathbb{M}$ of a dtMJLS is $p_{ij}(k) \triangleq \mathrm{Pr}(\omega : \theta_{k+1}(\omega) = j \mid \theta_k = i) = \mathrm{Pr}(\theta_{k+1} = j \mid \theta_k = i) \geq 0$. Since $p_{ij}(k)$ is a probability distribution $\forall i \in \mathbb{M}$, one has that the total mass of the distribution equals 1. Evidently, $\{\theta_k : k \in \mathbb{Z}_0\}$, with θ_k defined above, is a MC with TPM $P(k) \triangleq [p_{ij}(k)]$. The initial probability distribution of the MC is defined $\forall i \in \mathbb{M}$ by $p_i(0) \triangleq \mathrm{Pr}(\omega : \theta_0(\omega) = i) = \mathrm{Pr}(\theta_0 = i)$, and the initial probability distribution of all the operational modes is denoted by $\mathrm{p}_0 \triangleq [p_1(0), p_2(0), \ldots, p_N(0)]^T$. The expected value is denoted by $\mathbb{E}(\cdot)$. Following the line of reasoning of Costa et al. (2005), we set $\mathbb{H}^n \triangleq \mathbb{L}^2(\Omega, \mathcal{G}, \mathrm{Pr}, \mathbb{F}^n)$ the Hilbert space of all $\mathbb{F}^n$-valued $\mathcal{G}$-measurable random variables with inner product given $\forall \mathrm{x}, \mathrm{y} \in \mathbb{H}^n$ by $\langle \mathrm{x}, \mathrm{y} \rangle = \mathbb{E}(\mathrm{x}^* \mathrm{y})$, and Euclidean norm denoted by $\|\cdot\|_2$. Then, we set the direct sum of countably infinite copies of $\mathbb{H}^n$, denoted by $\ell_2(\mathbb{H}^n)$, which is a Hilbert space made up of collections of all $\mathbb{F}^n$-valued $\mathcal{G}$-measurable random variables indexed by the discrete time set $\mathbb{T} \triangleq \mathbb{Z}_0$, i.e., $\mathrm{f} = \{\mathrm{f}_k \in \mathbb{H}^n : k \in \mathbb{T}\}$ s.t. $\|\mathrm{f}\|_2^2 \triangleq \sum_{k=0}^{\infty} \mathbb{E}(\|\mathrm{f}_k\|^2) < \infty$. For each $\mathrm{f}, \mathrm{g} \in \ell_2(\mathbb{H}^n)$, the inner product is $\langle \mathrm{f}, \mathrm{g} \rangle \triangleq \sum_{k=0}^{\infty} \mathbb{E}(\mathrm{f}_k^* \mathrm{g}_k) \leq \|\mathrm{f}\|_2 \|\mathrm{g}\|_2$. Then, the space $\mathcal{H}^n \subseteq \ell_2(\mathbb{H}^n)$ is defined as follows. We say that $\mathrm{f} = \{\mathrm{f}_k \in \mathbb{H}^n : k \in \mathbb{T}\} \in \mathcal{H}^n$ if $\mathrm{f} \in \ell_2(\mathbb{H}^n)$ and $\mathrm{f}_k \in \mathbb{L}^2(\Omega, \mathcal{G}_k, \mathrm{Pr}, \mathbb{F}^n)$ $\forall k \in \mathbb{T}$. We have that $\mathcal{H}^n$ is a closed linear subspace of $\ell_2(\mathbb{H}^n)$ and therefore a Hilbert space (Costa et al., 2005, p. 21). We also define $\mathcal{H}_k^n$ as formed by sequences $(\mathrm{f}_t)_{t=0}^{k}$ s.t. $\mathrm{f}_t \in \mathbb{L}^2(\Omega, \mathcal{G}_t, \mathrm{Pr}, \mathbb{F}^n)$, $\forall t \in \mathbb{T}_k$, where the bounded discrete-time set $\mathbb{T}_k$ is defined as $\{t \in \mathbb{T} : t \leq k\}$. Finally, we denote by Θ_0 the set of all $\mathcal{G}_0$-measurable random variables taking values in $\mathbb{M}$. This permits to state the initial conditions for a dtMJLS with θ_k and x_k measurable $\forall k \in \mathbb{T}$ as $\mathrm{x}_0 \in \mathcal{H}_0^{n_\times}$, $\theta_0 \in \Theta_0$.

3   System model and stability definitions

Consider an autonomous dtMJLS described by the following state-space model defined on a stochastic basis

$(\Omega, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$, where x_k is a column vector of n_x either real or complex *state* variables, $\mathrm{v}_k \in \mathbb{F}^{n_\mathrm{v}}$ is a vector of *process noise* variables, $\boldsymbol{A} \triangleq (A_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{x}}$ is a sequence of *state matrices*, and $\boldsymbol{H} \triangleq (H_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{v}}$ is a sequence of *process noise matrices*, each of which is associated to an operational mode of the (switching) system; the values of $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$, i.e., $\boldsymbol{x}_0 \in \mathbb{F}^{n_\mathrm{x}}$ and $\vartheta_0 \in \mathbb{M}$, respectively, represent the initial conditions:

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + H_{\theta_k}\mathrm{v}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0, \ \theta_0 = \vartheta_0 \end{cases} \tag{1}$$

In this work we assume that the TPM $P(k)$ is varying over time, with variations that are arbitrary within a polytopic set of stochastic matrices. In order to express this statement formally, let $V \in \mathbb{Z}_+$ be a number of vertices of a convex polytope, and $\mathbb{V}$ be an index set of vertices of a convex polytope. Then, the set of vertices of a convex polytope of TPMs is formally defined as ${}_\mathbb{V}\mathbb{P} \triangleq \{P_l : l \in \mathbb{V}\}$. These vertices are obtained from measurement on the real system or via numerical reasoning, taking into account accuracy and precision of the measuring instruments and/or numerical algorithms. They bound the possible values each TP can assume. Then, the PTI assumption is stated as follows.

Assumption 1 *The time-varying TPM $P(k)$ is **polytopic**, i.e., $\forall k \in \mathbb{T}$, one has that*

$$P(k) = \sum_{l=1}^V \lambda_l(k) P_l, \quad \lambda_l(k) \geq 0, \quad \sum_{l=1}^V \lambda_l(k) = 1 \tag{2}$$

where $\forall l \in \mathbb{V}$, $P_l \in {}_\mathbb{V}\mathbb{P} \subset {}_\mathbb{V}\mathbb{R}^{N,N}$, i.e., P_l are elements of a given finite set of TPMs, which are the vertices of a convex polytope; moreover, $\lambda_l(k)$ are unmeasurable.

A visual representation of the concept of arbitrarily variations within a convex hull of points is illustrated in Figure 1, where P_1, P_2 and P_3 represent the vertices, and $P(k)$ shows a possible evolution in time of an element satisfying polytopic time-varying assumption.
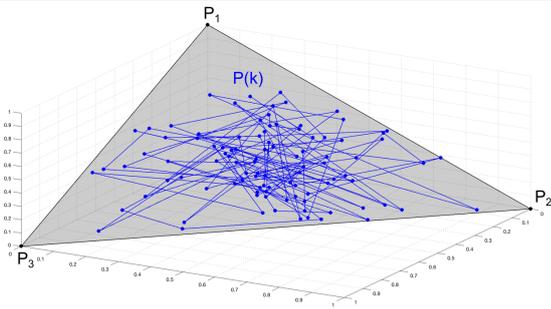

Fig. 1. Dynamics of an element satisfying Assumption 1

Following the standard workflow for dtMJLSs (Costa et al., 2005, p. 31), we use the indicator function to take advantage of the Markov property for system's state x_k,

and adopt the subsequent notation, where $k \in \mathbb{T}$ and $i \in \mathbb{M}$.

$$\mathrm{q}_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}^{n_\mathrm{x}} \tag{3}$$
$$\mathrm{q}(k) \triangleq [\mathrm{q}_1(k), \mathrm{q}_2(k), \ldots, \mathrm{q}_N(k)]^T \in \mathbb{F}^{Nn_\mathrm{x}}$$

$$\mathrm{r}_i(k) \triangleq \mathbb{E}\big(\mathrm{v}_k \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}^{n_\mathrm{v}} \tag{4}$$
$$\mathrm{r}(k) \triangleq [\mathrm{r}_1(k), \mathrm{r}_2(k), \ldots, \mathrm{r}_N(k)]^T \in \mathbb{F}^{Nn_\mathrm{v}}$$

$$Q_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}} \tag{5}$$

$$\boldsymbol{Q}(k) \triangleq (Q_i(k))_{i=1}^N \in {}_\mathrm{N}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}} \tag{6}$$

$$R_i(k) \triangleq \mathbb{E}\big(\mathrm{v}_k \mathrm{v}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}_0^{n_\mathrm{v}, n_\mathrm{v}} \tag{7}$$

$$\boldsymbol{R}(k) \triangleq (R_i(k))_{i=1}^N \in {}_\mathrm{N}\mathbb{F}_0^{n_\mathrm{v}, n_\mathrm{v}} \tag{8}$$

$$\boldsymbol{H}\boldsymbol{R}(k)\boldsymbol{H}^* \triangleq (H_i R_i(k) H_i^*)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}} \tag{9}$$

$$W_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k \mathrm{v}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}^{n_\mathrm{v}, n_\mathrm{x}} \tag{10}$$

$$\boldsymbol{W}(k) \triangleq (W_i(k)) \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{v}, n_\mathrm{x}} \tag{11}$$

$$\boldsymbol{A}\boldsymbol{W}(k)\boldsymbol{H}^* \triangleq (A_i W_i(k) H_i^*) \in {}_\mathrm{N}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}} \tag{12}$$

This permits to define the *expected value* of x_k as

$$\mathbb{E}(\mathrm{x}_k) = \sum_{i=1}^N \mathrm{q}_i(k) \in \mathbb{F}^{n_\mathrm{x}} \tag{13}$$

and the *second moment* of x_k as

$$\mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = \sum_{i=1}^N Q_i(k) \in \mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}} \tag{14}$$

The expressions of the first and second moment of x_k above can be easily derived from the definitions of the expected value and of the indicator function. This notation is used throughout the rest of the article to derive the necessary and sufficient conditions for (mean square) stability of PTI dtMJLSs as in (1).

The notion of stability of dtMJLSs that parallels the ideas of Lyapunov stability theory is the so-called mean square stability, which for a system described by (1) is defined as follows (Costa et al., 2005, pp. 36–37).

Definition 2 *A dtMJLS (1) is **MSS** if for any initial condition $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$, $\exists \boldsymbol{x}_e \in \mathbb{F}^{n_\mathrm{x}}$, $Q_e \in \mathbb{F}_+^{n_\mathrm{x}, n_\mathrm{x}}$ (independent from initial conditions x_0 and θ_0), such that*

$$\lim_{k \to \infty} \|\mathbb{E}(\mathrm{x}_k) - \boldsymbol{x}_e\| = 0, \tag{15a}$$

$$\lim_{k \to \infty} \|\mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) - Q_e\| = 0 \tag{15b}$$

Remark 3 *In noiseless case, i.e., when* $v_k = 0$ *in* (1), *the conditions* (15) *defining MSS become*

$$\lim_{k \to \infty} \mathbb{E}(x_k) = 0, \qquad \lim_{k \to \infty} \mathbb{E}(x_k x_k^*) = 0 \qquad (16)$$

as stated in Costa et al. (2005, p. 37, Remark 3.10).

There exist also other forms of stability for dtMJLSs without process noise, notably *exponential mean square stability* (EMSS) and *stochastic stability* (SS).

Definition 4 *A dtMJLS* (1) *is **EMSS** if for some reals* $\beta \geq 1$, $0 < \zeta < 1$, *we have for all initial conditions* $x_0 \in \mathcal{H}_0^{n_x}$ *and* $\theta_0 \in \Theta_0$ *that, for every* $k \in \mathbb{T}$, *if* $v_k = 0$, *then*

$$\mathbb{E}\left(\|x_k\|^2 \right) \leq \beta \zeta^k \|x_0\|_2^2 \qquad (17)$$

Definition 5 *A dtMJLS* (1) *is **stochastically stable** if for all initial conditions* $x_0 \in \mathcal{H}_0^{n_x}$ *and* $\theta_0 \in \Theta_0$, *we have that, if* $v_k = 0$ *for every* $k \in \mathbb{T}$, *then*

$$\sum_{k=0}^{\infty} \mathbb{E}\left(\|x_k\|^2 \right) < \infty \qquad (18)$$

Remark 6 *Clearly, in PTI setting, the conditions* (15) – (18) *are required to be satisfied for any possible sequence of values of* $P(k)$ *satisfying Assumption 1. Moreover,* (17) *points out the common maximal decay rate* ζ *for all possible evolutions of* $P(k)$. *Also, in time-homogeneous setting, the values of* $\boldsymbol{x}_e$ *and* Q_e *in* (15) *are the same for any initial starting time* k_0, *and any initial probability distribution* p_{k_0}. *This definition is used in the discussion of the stability of dtMJLSs affected by additive disturbance following the normal distribution, under the ergodic assumption for a stationary Markov chain (Costa et al., 2005, pp. 48–55). Unfortunately, an extension of this type of result is not possible in PTI setting, since an equivalent strong ergodic assumption (Isaacson & Madsen, 1976), guaranteeing convergence and loss of memory, would be too restrictive to be of any actual use. This is the reason why there is no discussion of the stability of the PTI dtMJLSs affected by additive process noise with a normal distribution in this article.*

In *time-homogeneous* case, i.e., when $P(k) = P \; \forall k \in \mathbb{T}$, there is a condition based on a value of a spectral radius of a matrix associated to the second moment of x_k that is necessary and sufficient for the (mean square) stability of a system described by (1); furthermore, in the noiseless case, MSS, EMSS and SS are equivalent (Costa et al., 2005, pp. 36–44, 48–49, 55–57). Specifically, the matrix related to the second moment of x_k that we have mentioned above is

$$\Lambda \triangleq \left(P^T \otimes I_{n_x^2} \right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}_i \otimes A_i \right) \right) \qquad (19)$$

The necessary and sufficient condition for the MSS of time-homogeneous dtMJLSs we have indicated before is

$\rho(\Lambda) < 1$. This condition for MSS does not hold in PTI case, as proven in Zacchia Lun et al. (2016), Zacchia Lun et al. (2017) and Lutz (2014, Remark 4.9), where all the considered time-homogeneous dtMJLSs have the values of the spectral radius of Λ smaller than 1, but when the PTI TPMs are allowed to switch arbitrarily in a set of TPMs at each time step, the resulting systems are unstable.

In Sections 5 and 6 we will present conditions, proven to be necessary and sufficient, for MSS of PTI dtMJLSs. The conditions are based on the generalization of the notion of spectral radius to sets of matrices. This generalization is known as joint spectral radius (or JSR).

4 Joint spectral radius

The JSR (Rota & Strang, 1960) is a generalization of the classical notion of spectral radius of a matrix, to sets of matrices. In the last decades JSR has been subject of intense research due to its role in the study of wavelets, switching systems, approximation algorithms, and many other topics (Jungers, 2009). In order to define JSR formally, let us denote by $_{\mathbb{V}}\mathbb{F}^{N,N}$ the set of all either real or complex square $N \times N$ matrices of cardinality V. For each $k \in \mathbb{Z}_+$, $\mathbb{P} \subseteq _{\mathbb{V}}\mathbb{F}^{N,N}$, let us consider the set $\mathcal{P}_k(\mathbb{P})$ of all possible products of length k whose factors are elements of $\mathbb{P}$, i.e.,

$$\mathcal{P}_k(\mathbb{P}) \triangleq \left\{ \left(\prod_{i=1}^{k} P_{l_i}^* \right)^* \in \mathbb{F}^{N,N} : P_{l_i} \in \mathbb{P} \subseteq _{\mathbb{V}}\mathbb{F}^{N,N} \right\} \quad (20)$$

For any matrix norm $\|\cdot\|$ on $\mathbb{F}^{N,N}$, consider the supremum among the normalized norms of all products in $\mathcal{P}_k(\mathbb{P})$, with $k \in \mathbb{Z}_+$, i.e., $\hat{\rho}_k(\mathbb{P}) \triangleq \sup_{P \in \mathcal{P}_k(\mathbb{P})} \|P\|^{\frac{1}{k}}$. Then, the *joint spectral radius* of $\mathbb{P} \subseteq _{\mathbb{V}}\mathbb{F}^{N,N}$ is defined as $\hat{\rho}(\mathbb{P}) = \lim_{k \to \infty} \hat{\rho}_k(\mathbb{P})$. The JSR of a bounded set of matrices has some interesting properties.

Proposition 7 *The convex hull of a set has the same joint spectral radius as the original set, i.e.,*

$$\hat{\rho}(\text{conv } \mathbb{P}) = \hat{\rho}(\mathbb{P}) \qquad (21)$$

PROOF. See Barabanov (1988). $\square$

Proposition 8 *For any bounded set of matrices* $\mathbb{P} \subseteq _{\mathbb{V}}\mathbb{F}^{N,N}$ *and for any* $k \in \mathbb{Z}_+$, *all matrix products* $P \in \mathcal{P}_k(\mathbb{P})$ *converge to zero matrix as* $k \to \infty$, *if and only if* $\hat{\rho}(\mathbb{P}) < 1$.

PROOF. See Berger & Wang (1992, Thm. I(b)). $\square$

These concepts are at the basis of our main results on (robust) stability of PTI dtMJLSs, presented next.

5 Stability conditions in noiseless setting

The results of this section are based on a noiseless version of (1), i.e., when $v_k = 0$ for every $k \in \mathbb{T}$:

$$\begin{cases} x_{k+1} = A_{\theta_k} x_k, \\ x_0 = \boldsymbol{x}_0, \ \theta_0 = \vartheta_0. \end{cases} \quad (22)$$

Let the TPM $P(k) = [p_{ij}(k)]$ of the system (22) be PTI, i.e., satisfying Assumption 1. Then, one can easily see that the recursive equations for $q_i(k)$ and $Q_i(k)$ defined by (3) and by (5), respectively, have the same structure as their counterpart in the time-homogeneous case with known probability matrix (Costa et al., 2005, p. 32):

Proposition 9 *In a dtMJLS* (22) *for all* $k \in \mathbb{T}, j \in \mathbb{M}$

$$q_j(k+1) = \sum\nolimits_{i=1}^{N} p_{ij}(k) A_i q_i(k) \quad (23)$$

$$Q_j(k+1) = \sum\nolimits_{i=1}^{N} p_{ij}(k) A_i Q_i(k) A_i^* \quad (24)$$

$$\mathbb{E}\left(\|x_k\|_2^2\right) = \mathbb{E}\left(\left\|\left(\prod\nolimits_{i=0}^{k-1} A_{\theta(i)}^*\right)^* x_0\right\|_2^2\right) \le n_x \|\boldsymbol{Q}(k)\|_1 \quad (25)$$

PROOF. See Appendix, Section A.2. $\square$

Similarly to the time-homogeneous case (Costa et al., 2005, pp. 33–35), also here, via application of Proposition 9 describing through (24) the dynamics of the matrices $Q_i(k)$, the definition of the linear operator $\mathrm{vec}^2(\cdot)$, and the related definition of the linear mapping $\mathrm{vec}(\cdot)$, the properties of the Kronecker product to $\boldsymbol{Q}(k)$, defined by (6), we have that $\mathrm{vec}^2(\boldsymbol{Q}(k+1)) = \Lambda(k)\mathrm{vec}^2(\boldsymbol{Q}(k))$, where $\Lambda(k)$ is a time-varying version of (19), i.e.,

$$\Lambda(k) \triangleq \left(P^T(k) \otimes I_{n_x^2}\right)\left(\bigoplus\nolimits_{i=1}^{N}\left(\bar{A}_i \otimes A_i\right)\right) \quad (26)$$

Proposition 10 $\Lambda(k)$ *is polytopic, i.e., for each* $k \in \mathbb{T}$

$$\Lambda(k) = \sum\nolimits_{l=1}^{V} \lambda_l(k)\Lambda_l, \ \lambda_l(k) \ge 0, \ \sum\nolimits_{l=1}^{V} \lambda_l(k) = 1, \quad (27a)$$

$$\Lambda_l \triangleq \left(P_l^T \otimes I_{n_x^2}\right)\left(\bigoplus\nolimits_{i=1}^{N}\left(\bar{A}_i \otimes A_i\right)\right), \quad (27b)$$

where for each $l \in \mathbb{V}$, $P_l \in {}_\mathbb{V}\mathbb{P} \subset {}_\mathbb{V}\mathbb{R}^{N,N}$, *i.e.,* P_l *are elements of a given finite set of TPMs, which are the vertices of a convex polytope.*

PROOF. See Appendix, Section A.3. $\square$

Similarly to ${}_\mathbb{V}\mathbb{P}$, let us indicate by ${}_\mathbb{V}\boldsymbol{\Lambda}$ the set of vertices of the convex polytope of the matrices $\Lambda(k)$ related to the second moment of x_k through $\boldsymbol{Q}(k)$. Then, $\forall k \in \mathbb{T}$ $\Lambda(k) \in \mathrm{conv}\,{}_\mathbb{V}\boldsymbol{\Lambda}$. It is worth noting that the set of possible values of $\Lambda(k)$ is bounded, but uncountable. Also, by Assumption 1, the values of $\Lambda(k)$ are unmeasurable.

Then, the repeated applications of (26) show that

$$\mathrm{vec}^2(\boldsymbol{Q}(k)) = \left(\prod\nolimits_{t=0}^{k-1} \Lambda^*(t)\right)^* \mathrm{vec}^2(\boldsymbol{Q}(0)) \quad (28)$$

The previous equation will be used in the proof of our first main result, presented in the next subsection.

5.1 Conditions for mean square stability

It is well known that the maximal rate of growth among all products of matrices from a bounded set is given by its JSR, that was introduced in Section 4 and will be used in the following theorem, which presents necessary and sufficient conditions for the MSS of PTI dtMJLSs.

Theorem 11 *A dtMJLS* (22) *with PTI TPM satisfying Assumption 1 is MSS if and only if* $\hat{\rho}({}_\mathbb{V}\boldsymbol{\Lambda}) < 1$.

PROOF. See Appendix, Section A.4. $\square$

The presented condition is very useful from a theoretical point of view, but it is computationally demanding, as shown in the next subsection. For additional details on the topic of computational complexity in general and NP-hardness in particular, see for instance Garey & Johnson (2002).

5.2 Computational complexity

While it is well known that the stability analysis problem for general switching systems (that is, deciding whether the JSR is smaller than 1) is NP-hard (Tsitsiklis & Blondel, 1997), we prove in the following theorem that it is NP-hard even in our particular model.

Theorem 12 *Given a dtMJLS* (22) *having PTI TPM satisfying Assumption 1, unless* $P = NP$, *there is no polynomial-time algorithm that decides whether it is mean square stable.*

PROOF. See Appendix, Section A.5. $\square$

Remark 13 *Although the computation of the exact value of the JSR is NP-hard, there exist efficient algorithms to approximate it up to any given accuracy (Protasov, Jungers & Blondel, 2010). Moreover, the stability*

problem is algorithmically decidable for sets of matrices that have the finiteness property (Jungers, 2009, Proposition 2.9), which holds if the set of matrices admits a complex polytope extremal norm (Guglielmi, Wirth & Zennaro, 2005). Then, the exact value of the JSR for the vast majority of matrix families in dimensions ≤ 20 can be found by the algorithm, that for nonnegative matrices works faster and finds the JSR in dimensions of order 100 within a few iterations. See Jungers, Cicone & Guglielmi (2014) for a sufficient condition for the existence of an extremal norm and efficient algorithms for computing the JSR, and Vankeerberghen, Hendrickx & Jungers (2014) for a description of a toolbox implementing the aforementioned algorithm. See also Ahmadi & Jungers (2016) for the connections between Lyapunov functions and the finiteness property of the optimal product that achieves the exact value of the JSR.

In the next subsection we present a theorem that links MSS to EMSS and to SS.

5.3 Stability equivalence

Our last but not least important result on stability of autonomous noiseless dtMJLSs as in (22) having PTI TPs is presented in the following theorem.

Theorem 14 *The following statements are equivalent.*

(1) The dtMJLS (22) is MSS;
(2) The dtMJLS (22) is EMSS;
(3) The dtMJLS (22) is SS.

PROOF. See Appendix, Section A.6. □

The results presented in this section, including several steps of the related proofs, are the basis for deriving the results of the next section, where on top of time-varying perturbations in uncertain TPMs, we consider also the presence of a bounded process noise.

6 Stability with bounded process noise

In order to better understand how the time-varying disturbances in uncertain TPMs affect the stability of dt-MJLSs, until now we have focused on state-space models without noise, control input, or any type of uncertainties in system parameters. Obviously, these parts of the model are not immune to the disturbances. Notably, the discrepancies between the modeled system states and the real process are often represented by an additive process noise, which in this section is described by an ℓ_2-stochastic signal. Such problem setup is particularly useful for the $\mathcal{H}_\infty$-control problems, as described by Costa et al. (2005, Chapter 7, pp. 143–166) for the dtMJLSs

with time-invariant and exactly known TPs between the operational modes.

Consider again an autonomous dtMJLS (1), where the TPM is time-varying, with variations that are arbitrary within a polytopic set, as formally stated by Assumption 1. The initial conditions are $x_0 \in \mathcal{H}_0^{n_x}$ and $\theta_0 \in \Theta_0$. It is easy to see by repeated applications of the recursion for x_k that the system state evolves as

$$x_k = \left(\prod_{t=0}^{k-1} A_{\theta_t}^*\right)^* x_0 + \sum_{t=0}^{k-1} \left(\prod_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t} v_t$$
$$= \check{x}_k + \sum_{t=0}^{k-1} \check{v}_t \qquad (29)$$

where the first addend $\check{x}_k$ on the right-hand side of the equality is clearly related to the noiseless version of system (1), while the other addend describes the contribution of the noise.

Let us indicate by $\mathrm{Re}[\cdot]$ either the real part of a complex number or, when applied to matrices, the operation of taking the real part of each entry of a complex matrix. As in the noiseless case, we find that the recursive equations for $q_i(k)$ and $Q_i(k)$ for PTI dtMJLSs as in (1) again have the same structure as their counterpart with time-homogeneous exactly known TPMs (Costa et al., 2005, pp. 50–52):

Proposition 15 *In a dtMJLS (1) for all $k \in \mathbb{T}$, $j \in \mathbb{M}$*

$$q_j(k+1) = \sum_{i=1}^{N} p_{ij}(k) A_i q_i(k) + \sum_{i=1}^{N} p_{ij}(k) H_i r_i(k) \quad (30)$$

$$Q_j(k+1) = \sum_{i=1}^{N} p_{ij}(k) A_i Q_i(k) A_i^* +$$
$$\sum_{i=1}^{N} p_{ij}(k) H_i R_i(k) H_i^* +$$
$$2\,\mathrm{Re}\left(\sum_{i=1}^{N} p_{ij}(k) A_i W_i(k) H_i^*\right) \quad (31)$$

PROOF. See Appendix, Section A.7. □

Following the same line as in the previous section, we rewrite the recursive equation (31) for $Q_i(k)$ in a matrix form. In particular, the recursive equation of $\boldsymbol{Q}(k)$ for dtMJLSs that accounts for a process noise is obtained by applying the equation (31) describing the dynamics of $Q_i(k)$ (from Proposition 15), together with the definition of the linear transformation $\mathrm{vec}^2(\cdot)$, the correspondent definition of the linear map $\mathrm{vec}(\cdot)$, and the relevant properties of the Kronecker product, to $\boldsymbol{Q}(k)$, defined by (6). Notably,

$$\mathrm{vec}^2(\boldsymbol{Q}(k+1)) = \Lambda(k)\mathrm{vec}^2(\boldsymbol{Q}(k)) + \Gamma(k)\mathrm{vec}^2(\boldsymbol{R}(k)) +$$
$$2\,\mathrm{Re}\big(\Xi(k)\mathrm{vec}^2(\boldsymbol{W}(k))\big) \quad (32)$$

where $\Lambda(k)$ is defined as in (26), $\boldsymbol{R}(k)$ is presented in (8), $\boldsymbol{W}(k)$ is shown in (11), and

$$\Gamma(k) \triangleq \left(P^T(k) \otimes I_{n_{\mathrm{x}}^2}\right)\left(\bigoplus_{i=1}^{N}\left(\bar{H}_i \otimes H_i\right)\right) \qquad (33)$$

$$\Xi(k) \triangleq \left(P^T(k) \otimes I_{n_{\mathrm{x}}^2}\right)\left(\bigoplus_{i=1}^{N}\left(\bar{H}_i \otimes A_i\right)\right) \qquad (34)$$

From the repeated applications of (32), we obtain

$$\mathrm{vec}^2\left(\boldsymbol{Q}(k)\right) = \left(\prod_{t=0}^{k-1}\Lambda^*(t)\right)^* \mathrm{vec}^2(\boldsymbol{Q}(0)) +$$
$$\sum_{t=0}^{k-1}\left(\prod_{j=t+1}^{k-1}\Lambda^*(j)\right)^* \Gamma(t)\mathrm{vec}^2(\boldsymbol{R}(t)) +$$
$$2\,\mathrm{Re}\left(\sum_{t=0}^{k-1}\left(\prod_{j=t+1}^{k-1}\Lambda^*(j)\right)^* \Xi(t)\mathrm{vec}^2(\boldsymbol{W}(t))\right) \quad (35)$$

Now we are ready to state the main result of this section. We will show that MSS for system (1) is equivalent to dtMJLS being a bounded linear operator that maps ℓ_2-stochastic exogenous input signals into ℓ_2-stochastic output signals.

Theorem 16 *Given a dtMJLS (1) with unknown time-varying TPM $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$, then $\hat{\rho}(_{\mathbb{V}}\Lambda) < 1$ if and only if $\mathrm{x} = \{\mathrm{x}_k : k \in \mathbb{T}\} \in \mathcal{H}^{n_{\mathrm{x}}}$ for every $\mathrm{v} = \{\mathrm{v}_k : k \in \mathbb{T}\} \in \mathcal{H}^{n_{\mathrm{v}}}$, and any initial condition $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$ and $\theta_0 \in \Theta_0$.*

PROOF. See Appendix, Section A.8. $\qquad\square$

As before, this result represents a useful generalization of the notion already known for time-homogeneous dt-MJLS. In fact, when there is only one time-invariant TPM, JSR corresponds to a spectral radius.

7 NASA F-8 Test Aircraft example

In this section, we present a practical example of dt-MJLS, in which the TPs are likely to be subject to uncertainties and time-varying characteristics. The data of this example from the aerospace industry is borrowed from Zhang, Yang, Shi & Zhu (2016, pp. 9–11) and references therein. We consider the NASA F-8 test aircraft at an attitude of 20000ft and a Mach number of 0.6. The discrete-time state-space model of the test aircraft in lateral-direction with the sampling time $T_s = 20$ ms is

$$\mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + H_{\theta_k}\mathrm{v}_k, \text{ with } \theta_k \in \mathbb{M} = \{i\}_{i=1}^{5},$$

where each possible setting is represented by a system mode, and for each $i \in \mathbb{M}$

$$A_i = e^{\mathcal{A}_i T_s}, \quad B_i = \int_0^{T_s} e^{\mathcal{A}_i T_s}\mathcal{B}_i, \quad H_i = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix}, \text{ with}$$

$$\mathcal{A}_1 = \begin{bmatrix} -2.6 & 0.25 & -38 & 0 & 17 & 7 \\ -0.075 & -0.27 & 4.4 & 0 & 0.82 & -3.2 \\ 0.078 & -0.99 & -0.23 & 0.052 & 0 & 0.046 \\ 1 & 0.078 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix},$$

$$\mathcal{A}_\star = \begin{bmatrix} 2.6 & 0.25 & 38 & 0 & 17 & 7 \\ 0.075 & 0.27 & 4.4 & 0 & 0.82 & 3.2 \\ 0.078 & 0 & 0.23 & 0 & 0 & 0.046 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{B}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_i & 0 \\ 0 & b_i \end{bmatrix},$$

$$\mathcal{A}_2 = \mathcal{A}_1 + 0.5\mathcal{A}_\star, \quad \mathcal{A}_3 = \mathcal{A}_1 - 0.5\mathcal{A}_\star,$$
$$\mathcal{A}_4 = \mathcal{A}_1 + 0.25\mathcal{A}_\star, \quad \mathcal{A}_5 = \mathcal{A}_1 - 0.25\mathcal{A}_\star,$$
$$a_1 = 20, \ b_1 = 20; \quad a_2 = 10, \ b_2 = 20; \quad a_3 = 20, \ b_3 = 10;$$
$$a_4 = 40, \ b_4 = 20; \quad a_5 = 20, \ b_5 = 50;$$
$$\mathrm{x}_k = \left[\gamma_r(k), \gamma_y(k), \alpha_s(k), \varsigma_r(k), \delta_a(k), \delta_r(k)\right]^T,$$
$$\mathrm{u}_k = \left[\delta_{ac}(k), \delta_{rc}(k)\right]^T,$$

where a_i and b_i are the dimensional lateral stability derivatives related to the incremental aileron and rudder positions with reference to the fixed body axes, δ_{ac} and δ_{rc} are the aileron and rudder servo commands (rad), γ_r and γ_y are the incremental roll and yaw rates (rad/s), α_s is the incremental sideslip angle (rad), ς_r is the incremental roll attitude (rad), while δ_a and δ_r are the incremental aileron and rudder positions (rad), respectively.

The nominal and perturbed values for the TPM are

$$P_n = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 & 0.4 \end{bmatrix}, P_p = \begin{bmatrix} 0.42 & 0.19 & 0.19 & 0.1 & 0.1 \\ 0.19 & 0.42 & 0.1 & 0.19 & 0.1 \\ 0.1 & 0.1 & 0.42 & 0.19 & 0.19 \\ 0.19 & 0.1 & 0.19 & 0.42 & 0.1 \\ 0.1 & 0.19 & 0.1 & 0.19 & 0.42 \end{bmatrix},$$

while the bounded process noise v_k is supposed to affect only the incremental roll and yaw rates, and the incremental sideslip angle, i.e.

$$\mathrm{v}_k = \left[\Delta\gamma_r(k), \Delta\gamma_y(k), \Delta\alpha_s(k)\right]^T, \text{ where } \forall k$$
$$\Delta\gamma_r \in [-0.035, 0.035], \ \Delta\gamma_y \in [-0.009, 0.009],$$
$$\Delta\alpha_s \in [-0.005, 0.005].$$

We observe that in this setting we have a typical situation for dtMJLSs, where there is an unstable operational mode, with $\rho(A_2) > 1$, but the switching system is stable (Costa et al., 2005, pp. 37–41). In particular, the autonomous dtMJLS is stable for both nominal and perturbed values of the TPM, with $\rho(\Lambda_n) = 0.999024$ and

$\rho(\Lambda_p) = 0.999023$. It should be noted that the size of both Λ is 180×180. Of course, the stability is guaranteed only when the transition probabilities are assumed to be time-invariant. From Theorem 12 and Remark 13, we would not expect to be able to decide the stability of the dt-MJLS through the computation of the JSR. However, by using the JSR toolbox (Vankeerberghen et al., 2014) we discover that the matrices Λ_n and Λ_p are triangularizable under a common permutation of the entries of the matrix (that is, their coordinate hyperplane is invariant), so that it is possible to prune the related sets of blocks. It turns out that the JSR can be computed in this case by considering two real matrices of size 80×80, and we apply the ellipsoid method based on semidefinite programming techniques to find that $\hat{\rho}(\{\Lambda_c, \Lambda_p\}) < 0.999112$, so the dtMJLS is stable even when the transition probabilities account for small time-varying perturbations, but the decay rate is very close to 1.

In order to improve the performance, one can e.g. design a state-feedback controller $u_k = K_{\theta_k} x_k$ that for the nominal transition probabilities provides a stabilizing solution for $\mathcal{H}_\infty$-control problem that takes into account the bounded process noise, as detailed in Costa et al. (2005, Chapter 7, pp. 143–166). For the system output $z_k = C_{\theta_k} x_k + D_{\theta_k} u_k$, with $C_i^* C_i = 0.0001 I_6$ and $D_i^* D_i = I_2$ for all $i \in \mathbb{M}$, and the disturbance attenuation level of 12, the obtained state-feedback control gain matrices are indicated by $(K_i)_{i=1}^5$. In the stability analysis of the controlled dtMJLS, the system state matrices A_i are substituted with $\hat{A}_i = A_i + B_i K_i$ in (27b), so that the new decay rates are bounded from below by $\rho(\hat{\Lambda}_n) = 0.992909$, and by $\rho(\hat{\Lambda}_p) = 0.992887$, respectively. Unfortunately, the matrices $\hat{\Lambda}_n$ and $\hat{\Lambda}_p$ are not jointly triangularizable, so the computation of the JSR involves the matrices of size 180×180, that are too big even for initialising the ellipsoid method on our test system (a MacBook Pro (retina, 13-inch, late 2013) with a 2.4 GHz dual-core Intel Core i5 processor, and 8 GB of 1600 MHz DDR3 RAM) within a week. However, if we considered NASA F-8 test aircraft to have just the first three operational modes and the system matrices described at the beginning of this section, with the values for the TPMs being for instance

$$\tilde{P}_n = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}, \tilde{P}_p = \begin{bmatrix} 0.48 & 0.26 & 0.26 \\ 0.22 & 0.50 & 0.28 \\ 0.22 & 0.28 & 0.50 \end{bmatrix},$$

the application of the related $\mathcal{H}_\infty$ state-feedback control gain matrices obtained in the same setting as before would produce $\rho(\tilde{\Lambda}_n) = 0.998702$, and $\rho(\tilde{\Lambda}_p) = 0.998695$, respectively. The size of new matrices would be 108×108, and the fact that $\hat{\rho}\left(\left\{\tilde{\Lambda}_c, \tilde{\Lambda}_p\right\}\right) < 0.999244$ could be established in less than 62 hours on our test system. For comparison, the computation of the bound on the JSR of the matrices of size 80×80 with the same accuracy requires around 8 hours.

With the constant growth of computational power and an intense research on the computation of the JSR by mathematical community, in the near future we expect to be able to solve the stability analysis problem for systems with the size (of matrices associated to the second moment of the system's state) larger than 100×100 in a much shorter amount of time.

8 Conclusions and future work

The conditions presented in this paper, based on the notion of the JSR of the set of vertices of the polytope of matrices characterizing the second moment of dtMJLS's state for all operational modes, permit to check whether the autonomous system is stable, regardless of the presence of bounded perturbations on the system's state itself. These results open up an unexplored research line on PTI dtMJLSs related to problems of robust linear quadratic regulation, optimal robust filtering, and separation of estimation and control.

A Appendix

It is useful to recall that for any $A \in \mathbb{F}^{n,n}$, the ℓ_1 and ℓ_2 norms satisfy the following inequality (Horn & Johnson, 2012, p. 365):

$$\|A\|_1 \leq n \|A\|_2 \qquad \text{(A.1)}$$

We remind that for positive semi-definite matrices, the trace dominates the ℓ_2 norm (De Klerk, 2002, p. 233):

$$\forall A \in \mathbb{F}_0^{n,n} \text{ one has that } \operatorname{tr}(A) \geq \|A\|_2 \qquad \text{(A.2)}$$

In the following subsections we present all the proofs of the results presented in this article. In particular, the derivation of the expressions for $q_i(k)$ and $Q_i(k)$ presented in Proposition 9 is straightforward, but the formal proof requires an additional lemma on inequality between trace of any positive semi-definite matrix and any matrix norm, which we present next.

Lemma 17 *For any $Q \in \mathbb{F}_0^{n,n}$ we have that*

$$\operatorname{tr}(Q) \leq n \|Q\| \qquad \text{(A.3)}$$

A.1 Proof of Lemma 17

The proof is based on the relationship between the trace, eigenvalues and the spectral radius $\rho(Q)$ of positive semi-definite matrices. Since Q is positive semi-definite, all its eigenvalues are nonnegative real numbers. Thus, from the definition of the absolute value for real numbers, the property of the trace of being the sum of all the eigenvalues of a square matrix, and definition of the spectral radius as the largest absolute value of the eigenvalues, we have that $\operatorname{tr}(Q) \leq n\rho(Q)$. Then, let $v \in \mathbb{F}^n$ be the

eigenvector associated to the maximal eigenvalue $\nu_{\max}$ of Q, which for both real and complex-valued positive semi-definite matrices equals to $\rho(Q)$. By definition of the eigenvalue, we have that $Q\mathrm{v} = \nu_{\max}\mathrm{v}$. By absolute homogeneity of any vector norm and triangle inequality of any matrix norm, we have for any $\nu_{\max} \in \mathbb{R}_0$ that

$$\nu_{\max} \|\mathrm{v}\| = |\nu_{\max}| \|\mathrm{v}\| = \|\nu_{\max}\mathrm{v}\| = \|Q\mathrm{v}\| \leq \|Q\| \|\mathrm{v}\|$$

Thus, $\rho(Q) = |\nu_{\max}| \leq \|Q\|$. Together with the first equation in the proof, this implies the thesis, and the lemma is proved. □

A.2 Proof of Proposition 9

The first statement can be proved by a sequential application of (3), (22), definition of the expected value and of the indicator function, linearity of expected value, definitions of the probability measure and the TP between the operational modes of a dtMJLS.

The second statement can be proved starting from the definition (5) of the matrix $Q_i(k)$, by following exactly the same considerations made for the first statement.

For what concerns (25), the equality, stated in the compact form of the matrix product in reverse order, comes from the repeated applications of the recursive equation (22) describing the evolution of the system's state x_k, while the inequality is derived by applying the definitions of the expected value and of the indicator function, of the trace and of the Euclidean norm, the linearity of the trace and of the expected value, (5), (A.3), triangle inequality, and definition of 1-norm for N-sequences of matrices. □

A.3 Proof of Proposition 10

The result follows from Assumption 1 on time-varying unmeasurable TPM $P(k)$ of being polytopic, by direct application of the related equation (2) and (bi-)linearity of the Kronecker product, to the definition (26) of the matrix $\Lambda(k)$. □

A.4 Proof of Theorem 11

We first prove the *necessity* of the presented condition for the MSS. By hypothesis (16), $\forall \mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$, $\lim_{k \to \infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = 0$.

First of all, we observe that the elements of the main diagonal of the positive semi-definite matrix $\mathrm{x}_k \mathrm{x}_k^*$ are all real and nonnegative. Formally, after indicating by $x_i(k)$ the i-th element of $\mathrm{x}_k \in \mathbb{F}^{n_\mathrm{x},1}$, by the definition of the conjugate transposition and the matrix multiplication,

we have that the elements of the main diagonal of the matrix $\mathrm{x}_k \mathrm{x}_k^*$ are

$$(x_i(k)\bar{x}_i(k))_{i=1}^{n_\mathrm{x}} \in \mathbb{R}_0^{n_\mathrm{x}} \tag{A.4}$$

where $\mathbb{R}_0^{n_\mathrm{x}}$ indicates an n_x-dimensional linear space, with entries in $\mathbb{R}_0$. From the definition (14) of the second moment of x_k and the definition (5) of $Q_i(k)$, we have

$$\lim_{k \to \infty} \sum\nolimits_{i=1}^{N} Q_i(k) = \lim_{k \to \infty} \sum\nolimits_{i=1}^{N} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}) = 0$$

Since limits of sequences behave well with respect to the usual arithmetic operations, we have that

$$\sum\nolimits_{i=1}^{N} \lim_{k \to \infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}) = 0$$

From the definition of the indicator function in a set of operational modes $\mathbb{M}$, considered together with (A.4), one has that, for each $i \in \mathbb{M}$ $\lim_{k \to \infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}) = \lim_{k \to \infty} Q_i(k) = 0$. Thus, from the definition (6) of $\boldsymbol{Q}(k)$ follows that

$$\lim_{k \to \infty} \boldsymbol{Q}(k) = 0 \tag{A.5}$$

The linear mapping $\mathrm{vec}^2(\cdot)$ is uniform homeomorphic (Costa et al., 2005, p. 17). As a consequence, the convergent behaviour of $\boldsymbol{Q}(k)$ is preserved by $\mathrm{vec}^2(\boldsymbol{Q}(k))$, i.e.,

$$\lim_{k \to \infty} \mathrm{vec}^2(\boldsymbol{Q}(k)) = 0$$

Applying (28) for the recursion of $\mathrm{vec}^2(\boldsymbol{Q}(k))$, we obtain

$$\lim_{k \to \infty} \left(\prod\nolimits_{t=0}^{k-1} \Lambda^*(t) \right)^* \mathrm{vec}^2(\boldsymbol{Q}(0)) = 0 \tag{A.6}$$

From Proposition 10, we have that $\Lambda(k) \in \mathrm{conv}_\mathbb{V} \boldsymbol{\Lambda}$ for each $k \in \mathbb{T}$. Thus, from Proposition 8, we have that (A.6) holds for any $\boldsymbol{Q}(0)$ if and only if $\hat{\rho}(\mathrm{conv}_\mathbb{V}\boldsymbol{\Lambda}) < 1$. Then, from Proposition 7 follows the thesis.

Now, let us prove that the presented condition is indeed sufficient, by showing that the MSS of system (22) is implied by $\hat{\rho}(\mathbb{V}\boldsymbol{\Lambda}) < 1$. As the definition of MSS (16) provides two requirements, one for the expected value, and other for the second moment of the system's state x_k, for k approaching infinity, the proof of sufficiency is divided in two parts.

The first part of the proof follows the inverse pattern of the proof of the necessity. We start with the expression (28) for the recursion of $\mathrm{vec}^2(\boldsymbol{Q}(k))$. By its definition, provided by (6) and (5), $\boldsymbol{Q}(0)$ accounts for all possible initial operational modes $\theta_0 \in \mathbb{M}$; it depends only on the initial state x_0 of the system, and the initial probability distribution p_0 of all the operational modes. Thus, there always exists a $\boldsymbol{Q}(0) \in {}_\mathrm{N}\mathbb{F}_0^{n_\mathrm{x},n_\mathrm{x}}$ for any initial condition, represented by the values of $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$.

Besides, the matrix $\boldsymbol{Q}(0)$ accounts also for any initial probability distribution p_0. Since, by Proposition 10, $\Lambda(k) \in \mathrm{conv}_{\mathbb{V}}\boldsymbol{\Lambda} \ \forall k \in \mathbb{T}$, we have that

$$\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}) < 1 \Rightarrow \lim_{k \to \infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = 0, \quad \forall \mathrm{x}_0 \in \mathcal{H}_0^{n_x}, \ \theta_0 \in \Theta_0$$

by Propositions 7 and 8, uniform homeomorphism between the spaces $_{\mathrm{N}}\mathbb{F}_0^{n_x, n_x}$ and $\mathbb{F}_0^{N n_x^2}$ through the mapping $\mathrm{vec}^2(\cdot)$, together with the application of the definitions of matrices $\boldsymbol{Q}(k)$, $Q_i(k)$ and of the second moment of x_k, i.e., (6), (5) and (14), respectively.

To complete the proof, in this second part of the proof of sufficiency we need to show that $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}) < 1$ implies that $\lim_{k \to \infty} \mathbb{E}(\mathrm{x}_k) = 0$, $\forall \mathrm{x}_0 \in \mathcal{H}_0^{n_x}$, $\theta_0 \in \Theta_0$. From the first part of the proof of sufficiency, we already have (A.5), i.e., that the matrix $\boldsymbol{Q}(k)$ converges to the zero matrix as $k \in \mathbb{T}$ approaches infinity. Then, the equation (25) from Proposition 9 tells us that the value of the expected value of $\|\mathrm{x}_k\|_2^2$ is bounded by $\|\boldsymbol{Q}(k)\|_1$. Thus, we obtain that $\lim_{k \to \infty} \mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) = 0$. Since limits of sequences behave well with respect to the usual arithmetic operations, including multiplication, and thus, exponentiation, we have that $\lim_{k \to \infty} \mathbb{E}(\|\mathrm{x}_k\|_2) = 0$, which implies the thesis and concludes the proof. $\square$

A.5 Proof of Theorem 12

Let us indicate by $\mathbb{Q}^{n,n}$ the set of all square matrices of order n with entries in $\mathbb{Q}$. Our proof works by reduction from the *matrix semigroup stability*, which is well known to be NP-hard (Jungers, 2009, Theorem 2.4 and Theorem 2.6). In this problem, one is given a set of two matrices $\mathbb{S}_{\mathrm{M}} = \{\mathrm{M}, \mathrm{M}'\} \subset \mathbb{Q}^{n,n}$ such that $\mathrm{M} = [m_{ij}]$, $\mathrm{M}' = [m'_{ij}]$, and for any $i, j \in \mathbb{Z}_+$, $m_{ij} \in \mathbb{Q}_0$ and $m'_{ij} \in \mathbb{Q}_0$, i.e., the entries of the two square matrices of order n are all nonnegative rational numbers. Then, one is asked whether the product of length k of any sequence of matrices M, M' converges to the zero matrix when $k \to \infty$.

Let us consider a particular instance of the matrix semigroup stability problem. We will build an autonomous noiseless dtMJLS (22) with a set of (scalar) state matrices $\{a_i \in \mathbb{R}_0 : i, n, N \in \mathbb{Z}_+, \ N = n+1, \ i \leq N\}$, with unknown and time-varying TPMs $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$, with $_{\mathbb{V}}\mathbb{P} = \{P, P'\} \subset \mathbb{R}^{N,N}$, where $P = [p_{ij}]$ and $P' = [p'_{ij}]$ are stochastic matrices (i.e., for any $i, j \in \mathbb{Z}_+$, $p_{ij} \in \mathbb{R}_0$, $p'_{ij} \in \mathbb{R}_0$, and such that any row of these two matrices is a distribution) and prove that the constructed dtMJLS (22) is MSS if and only if the set $\mathbb{S}_{\mathrm{M}}$ is stable. By (27) it follows that $\Lambda = P^T \left(\bigoplus_{i=1}^N a_i^2\right)$, $\Lambda' = (P')^T \left(\bigoplus_{i=1}^N a_i^2\right)$, and our construction is as follows. Assign arbitrarily for each $j \in \mathbb{Z}_+$, s.t. $j \leq n$, $a_j^2 \in \mathbb{Q}_0 : a_j^2 \geq n \max_i \{m_{ij}, m'_{ij} : i \in \mathbb{Z}_+, i \leq n\}$. Assign for all $i, j \in \mathbb{Z}_+$ s.t. $i \leq n$, $j \leq n$ $p_{ij} \triangleq \frac{m_{ji}}{a_i^2}$, $p'_{ij} \triangleq \frac{m'_{ji}}{a_i^2}$. Obviously, $p_{ij}, p'_{ij} \in \mathbb{Q}_0$, $p_{ij} \leq \frac{1}{n}$,

and $p'_{ij} \leq \frac{1}{n}$. Then, assign for every $i \in \mathbb{Z}_+$ s.t. $i \leq N$ $p_{iN} \triangleq 1 - \sum_{j=1}^n p_{ij}$, $p'_{iN} \triangleq 1 - \sum_{j=1}^n p'_{ij}$. Clearly, by construction $p_{iN}, p'_{iN} \in \mathbb{Q}_0$, $p_{iN} \leq 1$, $p'_{iN} \leq 1$. As a next step, assign $a_N \triangleq 0$. Finally, for each $j \in \mathbb{Z}_+$, $j \leq N$ assign $p_{Nj} = p'_{Nj} \triangleq \frac{1}{N}$. As a consequence of the above assignments, it follows that P, P' are stochastic matrices and that

$$\Lambda = \begin{bmatrix} \mathrm{M} & 0 \\ \mathrm{R} & 0 \end{bmatrix}, \qquad \Lambda' = \begin{bmatrix} \mathrm{M}' & 0 \\ \mathrm{R}' & 0 \end{bmatrix},$$

with $\mathrm{R}, \mathrm{R}' \in \mathbb{Q}^{1,n}$, having nonnegative elements. By Theorem 11, (22) is MSS if and only if the JSR of the set $\{\Lambda, \Lambda'\}$ is smaller than 1. From this, it is straightforward to see that (22) is MSS if and only if $\mathbb{S}_{\mathrm{M}}$ is stable. This concludes the proof. $\square$

Remark 18 *It is not known (to the best of our knowledge) whether the matrix semigroup stability problem is Turing decidable (say, for matrices with rational nonnegative entries). Thus, the above proof does not allow us to conclude that MSS is undecidable for dtMJLSs with polytopic unknown and time-varying TPMs. This is why we only claim that the stability problem is NP-hard.*

A.6 Proof of Theorem 14

In this article we are working on a finite-dimensional linear spaces, for which all norms are equivalent. Thus, in the following proof we will make use of Euclidean and grid norms for vectors, ℓ_1 and ℓ_2 norms for matrices, and 1-norm for sequences of matrices. We invite an interested reader to see Horn & Johnson (2012, Section 5.6, pp. 340–370) for a detailed presentation of the topic of matrix norms, especially as a reference for the exact constants used in the inequalities involving the equivalent norms.

It is trivially verified that the second assertion in the statement of the theorem implies the third one, i.e., EMSS $\Rightarrow$ SS. The result follows directly from the definitions of EMSS (17) and SS (18).

Thus, let us show that the third statement implies the first, that is, SS $\Rightarrow$ MSS. We have already seen in the proof of (25) in Proposition 9 that from the definition of the Euclidean norm for vectors and the definition of trace as a linear mapping (together with the definition of the matrix product and linearity of the expected value), one obtains that

$$\mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) = \mathrm{tr}(\mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*)) = \mathbb{E}(\mathrm{x}_k^* \mathrm{x}_k) \geq 0 \qquad (\mathrm{A}.7)$$

Then, the absolute convergence of a series in a normed linear space implies the convergence of a series in the same space. So, from the definition of the stochastic sta-

bility (18) and (A.7), one has that, for all initial conditions $x_0 \in \mathcal{H}_0^{n_x}$ and $\theta_0 \in \Theta_0$

$$\lim_{k \to \infty} \operatorname{tr}(\mathbb{E}(x_k x_k^*)) = \lim_{k \to \infty} \mathbb{E}(x_k^* x_k) = 0$$

As a consequence (since $x_k x_k^*$ defines a positive semidefinite matrix, for which, by Horn & Johnson (2012, Corollary 7.1.5, p. 430), $\operatorname{tr}(x_k x_k^*) = 0$ if and only if $x_k x_k^* = 0$), this implies $\lim_{k \to \infty} \mathbb{E}(x_k x_k^*) = 0$.

We have already seen in the proof of the sufficiency of Theorem 11 that this last statement implies MSS of the system (22). Hence, this part of the proof is concluded. Moreover, as a result, we have also that EMSS $\Rightarrow$ MSS.

Now, let us show that the opposite is true as well, that is, MSS $\Rightarrow$ EMSS. From Theorem 11 we know that if the system (22) is MSS, then $\hat{\rho}(\mathbb{V}\boldsymbol{\Lambda}) < 1$. Since from the definition of the JSR

$$\lim_{k \to \infty} \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\|^{\frac{1}{k}} \leq \hat{\rho}(\mathbb{V}\boldsymbol{\Lambda})$$

by the *radical test* (a.k.a. *Cauchy root test*) for infinite series we state that, for some integer $k' \geq 0$

$$\left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\| < \zeta^k, \ \forall k \geq k', \ \forall \zeta \in \mathbb{R}_+ : \zeta \in (\hat{\rho}(\mathbb{V}\boldsymbol{\Lambda}), 1),$$

$$\beta' = \zeta^{-k'} \sup_{\mathrm{P} \in \mathcal{P}_j(\mathbb{V}\boldsymbol{\Lambda}), \, 0 \leq j \leq k'} \|\mathrm{P}\|, \quad \beta' \geq 1,$$

where $\mathcal{P}_j(\mathbb{V}\boldsymbol{\Lambda})$ indicates the set of all possible products of length j whose factors are elements of $\mathbb{V}\boldsymbol{\Lambda}$, as formally defined by (20). So, we obtain that

$$\left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\| \leq \beta' \zeta^k, \quad \forall k \in \mathbb{T} \qquad (A.8)$$

Now, in the proof of (25) in Proposition 9 we have seen that

$$\mathbb{E}\left(\|x_k\|_2^2 \right) \leq n_x \sum_{i=1}^{N} \|Q_i(k)\| \qquad (A.9)$$

To proceed with our proof, we use the ℓ_1-norm as the particular norm for $Q_i(k)$, which we are going to examine next. We indicate by $\Lambda_{[(i-1)n_x^2+1, in_x^2]\bullet}$ a matrix obtained by taking n_x consecutive rows (starting from the $((i-1)n_x^2 + 1)$-th row, with $i \in \mathbb{Z}_+$, $i \leq N$) of Λ. From the recursion (28) for $\boldsymbol{Q}(k)$, by using the linear mapping $\operatorname{vec}^2(\cdot)$, the definitions of the ℓ_1-norm, the matrix product, the sub-multiplicative property of matrix norms and

triangle inequality, we have that

$$
\begin{aligned}
\|Q_i(k)\|_1 &= \|\operatorname{vec}(Q_i(k))\|_1 \\
&= \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^*_{[(i-1)n_x^2+1, in_x^2]\bullet} \operatorname{vec}^2(\boldsymbol{Q}(0)) \right\|_1 \\
&\leq \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\|_1 \|\operatorname{vec}^2(\boldsymbol{Q}(0))\|_1 \qquad (A.10)
\end{aligned}
$$

since the matrix norm of a submatrix is always less than or equal to a norm of the whole matrix.

Since (A.10) is valid for each $i \in \mathbb{M}$, we rewrite (A.9) as

$$\mathbb{E}\left(\|x_k\|_2^2 \right) \leq n_x N \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\|_1 \|\operatorname{vec}^2(\boldsymbol{Q}(0))\|_1 \quad (A.11)$$

From the definition of the ℓ_1-norm, it is trivial to see that

$$\|\operatorname{vec}^2(\boldsymbol{Q}(0))\|_1 = \sum_{i=1}^{N} \|\operatorname{vec}(Q_i(0))\|_1 = \sum_{i=1}^{N} \|Q_i(0)\|_1 \quad (A.12)$$

At this point, we recall that the ℓ_1-norm is related to the ℓ_2-norm by the inequality (A.1), i.e.,

$$\|Q_i(0)\|_1 \leq n_x \|Q_i(0)\|_2 \qquad (A.13)$$

Since by its definition (5), $Q_i(0) \in \mathbb{F}^{n_x, n_x}$ for each $i \in \mathbb{M}$, we apply the property (A.2) of the trace of a positive semi-definite matrix, obtaining that

$$\|Q_i(0)\|_2 \leq \operatorname{tr}(Q_i(0)) \qquad (A.14)$$

After combining (A.11) with (A.12), (A.13), and (A.14), we obtain

$$\mathbb{E}\left(\|x_k\|_2^2 \right) \leq n_x^2 N \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t) \right)^* \right\|_1 \sum_{i=1}^{N} \operatorname{tr}(Q_i(0)) \quad (A.15)$$

Within the proof of (25) in Proposition 9, we have seen that (with $k = 0$)

$$\sum_{i=1}^{N} \operatorname{tr}(Q_i(0)) = \mathbb{E}\left(\|x_0\|_2^2 \right) = \|x_0\|_2^2 \qquad (A.16)$$

Then, by putting together (A.15), (A.16), and (A.8), we obtain

$$\mathbb{E}\left(\|x_k\|_2^2 \right) \leq n_x^2 N \beta' \zeta^k \|x_0\|_2^2 = \beta \zeta^k \|x_0\|_2^2$$

This proves the assertion that MSS $\Rightarrow$ EMSS also for the dtMJLSs (22) with time-varying uncertain TPs. All the remaining implications follow from the already proved ones. Thus, the proof is concluded. □

The proof is very similar to the proof of Proposition 9, so we only outline the procedure.

The first statement is obtained from the definition (3) of the vector $q_i(k)$ of expected values of the system state variables in correspondence of the i-th operational mode, the recursive equation (1) describing the evolution of the system's state x_k, the definition (4) of the vector $r_i(k)$ of expected values of the process noise related to the i-th operational mode, with $i \in \mathbb{M}$, by linearity of the expected value.

The second statement can be proven in the same manner, i.e., by linearity of the expected value, from the definition (5) of the matrix $Q_i(k)$, the state-space representation (1) of the dtMJLS, the definition (7) of the matrix $R_i(k)$, and the definition (10) of the matrix $W_i(k)$, after remembering the properties of complex conjugation, and the fact that the sum of a complex number with its complex conjugate gives us two times the real part of the complex number. $\square$

A.8   Proof of Theorem 16

To prove the **necessity** (i.e., $\hat{\rho}(_\mathbb{V}\mathbf{\Lambda}) < 1 \Rightarrow x \in \mathcal{H}^{n_x}$ $\forall v \in \mathcal{H}^{n_v}$, $x_0 \in \mathcal{H}_0^{n_x}$, $\theta_0 \in \Theta_0$), all we have to show is that $\|x\|_2 < \infty$ since on the considered stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_k), \text{Pr})$ we clearly have that $(x_t)_{t=0}^k \in \mathcal{H}_k^{n_x}$ for each $k \in \mathbb{T}$. We start by looking at the equation (29) describing x_k as a function of x_0, noticing that the first addend $\check{x}_k$ on the right-hand side of the equality is clearly related to the noiseless version of system (1). The other addend describes the contribution of the noise. So, the proof of necessity is divided into three parts: the first one is related to the noiseless part of the evolution of the system's state $\check{x}_k$, the second part is connected to the partial dynamics due to the noise, i.e., $\check{v}_t$, while the last part corresponds to the combination of the previous two.

By hypothesis $\hat{\rho}(_\mathbb{V}\mathbf{\Lambda}) < 1$, and for the first part we apply the same steps of the proof of the fact that, for dtMJLS without process noise, EMSS implies MSS. This procedure is illustrated in the proof of Theorem 14, where it is shown that for some $k' \in \mathbb{T}$, there always exists a $k \in \mathbb{T}$, $k \geq k'$, such that for any $N, n_x \in \mathbb{Z}_+$,

$$\|\check{x}_k\|_2^2 = \mathbb{E}\left(\|\check{x}_k\|_2^2\right) \leq \beta\zeta^k\|x_0\|_2^2 = Nn_x^2\beta'\zeta^k\|x_0\|_2^2 \quad \text{(A.17)}$$

with $\zeta \in \mathbb{R}_+$ such that $\qquad \zeta \in (\hat{\rho}(_\mathbb{V}\mathbf{\Lambda}), 1) \qquad$ (A.18)

and $\beta' \in \mathbb{R}_+$, $\beta' \geq 1$, being defined as

$$\beta' = \zeta^{-k'} \sup_{P \in \mathcal{P}_j(_\mathbb{V}\mathbf{\Lambda}), \, 0 \leq j \leq k'} \|P\| \qquad \text{(A.19)}$$

where $\mathcal{P}_j(_\mathbb{V}\mathbf{\Lambda})$ indicates the set of all possible products of length j whose factors are elements of $_\mathbb{V}\mathbf{\Lambda}$, as formally defined by (20).

Regarding the second part, which is related to the evolution in time of the partial contribution of the noise, the procedural steps are similar.

For each $t \in \mathbb{T}_{k-1}$ we consider $\check{v}_t = \left(\prod_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t} v_t$. It is clear from the previous expression that $\check{v}_t$ behaves as an autonomous noiseless dtMJLS with the initial condition given by $H_{\theta_t} v_t$. The second moment for this initial condition, i.e., $\mathbb{E}\left(H_{\theta_t} v_t \left(H_{\theta_t} v_t\right)^*\right)$, equals

$$\sum_{i=1}^N \mathbb{E}\left(H_{\theta_t} \left(v_t v_t^* \mathbf{1}_{\{\theta_t = i\}}\right) H_{\theta_t}^*\right) = \sum_{i=1}^N H_i R_i(t) H_i^* p_i(t)$$

So, it is trivial to verify that the second moment of $\check{v}_t$ is already computed in matrix form in the equation (35) describing the evolution of the second moment of the autonomous system (1) with a process noise. After defining $\check{R}_i(t) \triangleq \mathbb{E}\left(\check{v}_t \check{v}_t^* \mathbf{1}_{\{\theta_t = i\}}\right) \in \mathbb{F}_0^{n_x, n_x}$ and $\check{\boldsymbol{R}}(t) \triangleq \left(\check{R}_i(t)\right)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}_0^{n_x, n_x}$, and recalling that $\Gamma(t)$ is given by (33), we obtain that

$$\mathrm{vec}^2\left(\check{\boldsymbol{R}}(t)\right) = \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \Gamma(t) \mathrm{vec}^2(\boldsymbol{R}(t))$$
$$= \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)$$

where $\boldsymbol{R}(t)$ and $R_i(t)$ are expressed via (8) and (7), respectively, and $\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*$ is represented by (9). From here on, we follow the line of the proof of Theorem 14. Thus, we only outline the main points, without explaining every passage. First of all, we make the same considerations used in the proof of the third statement (25) in Proposition 9, especially the inequality (A.3) between the trace of a matrix and any norm of the same matrix, proved in Lemma 17, obtaining that $\forall t \in \mathbb{T}_{k-1}$

$$\|\check{v}_t\|_2^2 = \mathbb{E}\left(\|\check{v}_t\|_2^2\right) = \sum_{i=1}^N \mathrm{tr}\left(\check{R}_i(k)\right) \leq n_x \sum_{i=1}^N \|\check{R}_i(k)\|$$

which holds for any equivalent matrix norm, including ℓ_1-norm. We apply ℓ_1-norm to $\check{R}_i(k)$, obtaining that

$$\|\check{R}_i(k)\|_1 = \left\|\left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^*_{[(i-1)n_x^2+1, in_x^2]\bullet} \mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\right\|_1$$
$$\leq \left\|\left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)\right\|_1 \|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\|_1$$

Since the previous inequality is valid $\forall i \in \mathbb{M}$, we write

$$\|\check{v}_t\|_2^2 \leq n_x N \left\|\left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)\right\|_1 \|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\|_1 \quad \text{(A.20)}$$

From the definition of the ℓ_1-norm, it follows that

$$\left\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\right\|_1 = \sum_{i=1}^{N}\|H_i R_i(t)H_i^*\|_{\mathbf{1}}$$

Recollecting that the ℓ_1-norm is related to the ℓ_2-norm by the inequality (A.1), we have that

$$\|H_i R_i(t)H_i^*\|_{\mathbf{1}} \leq n_{\mathrm{x}}\,\|H_i R_i(t)H_i\|_{\mathbf{2}}$$

As before, by construction $H_i R_i(t)H_i^* \in \mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}}$ for each $i \in \mathbb{M}$. Thus, we apply the property (A.2) of the trace of a positive semi-definite matrix, together with the definition (7) of $R_i(t)$, obtaining that

$$\|H_i R_i(t)H_i^*\|_{\mathbf{2}} \leq \mathrm{tr}\big(H_i \mathbb{E}\big(\mathrm{v}_t \mathrm{v}_t^* \mathbf{1}_{\{\theta_t = i\}}\big)\,H_i^*\big)$$

Consequently, from the previous three equations, by linearity of the trace and definition of the max-norm on the linear space made up of all N-sequences of either real or complex matrices, we obtain that

$$\left\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\right\|_1 \leq n_{\mathrm{x}}\,\|\boldsymbol{H}\|_{\mathrm{max}}^2\,\|\mathrm{v}_t\|_2^2$$

Then, by the radical test for infinite series, valid for all equivalent matrix norms, we also have that

$$\left\|\left(\prod_{j=t+1}^{k-1}\Lambda^*(j)\right)\right\|_{\mathbf{1}} \leq \beta'\zeta^{k-t-1}$$

where ζ and β' are those defined by (A.18) and (A.19), respectively. Putting together both parts of (A.20), we obtain that

$$\|\check{\mathrm{v}}_t\|_2^2 \leq N n_{\mathrm{x}}^2 \beta'\zeta^{k-i-1}\,\|\boldsymbol{H}\|_{\mathrm{max}}^2\,\|\mathrm{v}_t\|_2^2 \qquad \text{(A.21)}$$

which holds for each $t \in \mathbb{T}_{k-1}$. So, we have (29) together with bounds on $\|\check{\mathrm{x}}_k\|_2^2$ expressed by (A.17) and on $\|\check{\mathrm{v}}_t\|_2^2$, given by (A.21). By triangle inequality, we have that

$$\|\mathrm{x}_k\|_2 \leq \|\check{\mathrm{x}}_k\|_2 + \sum_{t=0}^{k-1}\|\check{\mathrm{v}}_t\|_2\,.$$

We still need to show that $\|\mathrm{x}_k\|_2 < \infty$. From now on, in this last part of the proof of the necessity, we follow the steps of the proof provided in Costa et al. (2005, Theorem 3.34, pp. 55-57) for time-homogeneous dtMJLSs with bounded process noise. Applying the bounds obtained for $\|\check{\mathrm{x}}_k\|_2^2$ in (A.17) and for $\|\check{\mathrm{v}}_t\|_2^2$ in (A.21), and also considering the expressions (A.18), (A.19) of respectively ζ and β', we can state that there exist $\zeta \in (\hat{\rho}(\mathbb{V}\boldsymbol{\Lambda}),1)$ and $\beta' \geq 1$ s.t. $\|\mathrm{x}_k\|_2 \leq \sum_{t=0}^{k}\zeta_{k-t}\beta_t$, with $\zeta_{k-t} \triangleq \sqrt{\zeta^{k-t}}$, $\beta_t \triangleq n_{\mathrm{x}}\sqrt{N\beta'}\|\boldsymbol{H}\|_{\mathrm{max}}\|\mathrm{v}_{t-1}\|_2$ for $t \geq 1$, and $\beta_0 \triangleq n_{\mathrm{x}}\sqrt{N\beta'}\,\|\mathrm{x}_0\|_2$. We set $a \triangleq (\zeta_i)_{i=0}^{k}$ and $b \triangleq (\beta_i)_{i=0}^{k}$. Since $a \in \ell_1$ (i.e. $\sum_{i=0}^{\infty}|\zeta_i| < \infty$) and $b \in \ell_2$ (that is, $\sum_{i=0}^{\infty}|\beta_i|^2 < \infty$), it follows that the convolution $c \triangleq a * b = (c_i)_{i=0}^{k}$, $c_i \triangleq \sum_{t=0}^{i}\zeta_{i-t}\beta_t$, lies itself in ℓ_2

with $\|c\|_2 \leq \|a\|_1\|b\|_2$ (see e.g. Costa et al. (2005, p. 56)). Hence,

$$\|\mathrm{x}\|_2 = \sqrt{\sum_{k=0}^{\infty}\mathbb{E}\big(\|\mathrm{x}_k\|_2^2\big)} \leq \sqrt{\sum_{i=0}^{\infty}c_i^2} = \|c\|_2 < \infty$$

This concludes the proof of necessity.

To prove the **sufficiency** (that is, $\mathrm{x} \in \mathcal{H}^{n_{\mathrm{x}}}$ $\forall\,\mathrm{v} \in \mathcal{H}^{n_{\mathrm{v}}}$, $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$, $\theta_0 \in \Theta_0 \Rightarrow \hat{\rho}(\mathbb{V}\boldsymbol{\Lambda}) < 1$) we observe that, by hypothesis, $\|\mathrm{x}\|_2^2 = \sum_{k=0}^{\infty}\mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) < \infty$ for all $\mathrm{v} \in \mathcal{H}^{n_{\mathrm{v}}}$, $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$, and $\theta_0 \in \Theta_0$. Since the absolute convergence of a series in a normed linear space implies the convergence of a series in that space, by Megginson (1998, Proposition 1.3.7) we have that $\lim_{k \to \infty}\mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = 0$ for all $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$, $\theta_0 \in \Theta_0$, and for any $\mathrm{v} \in \mathcal{H}^{n_{\mathrm{v}}}$. Since this last statement holds for every $\mathrm{v} \in \mathcal{H}^{n_{\mathrm{v}}}$, we can make $\mathrm{v}_k = 0$ for all $k \in \mathbb{T}$ in the state-space representation of the autonomous dt-MJLS (1), obtaining the noiseless model (22). Thus, we have exactly the same conditions found at the beginning of the proof of necessity of Theorem 11. Application of the procedure illustrated there brings us to the thesis and concludes our proof of sufficiency. $\quad\square$

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