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On stability of time-inhomogeneous Markov jump linear systems

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***Abstract*— In this work we present necessary and sufficient conditions for mean square stability (MSS) of discrete-time time-inhomogeneous Markov jump linear systems (MJLS) affected by polytopic uncertainties on transition probabilities. We also prove that deciding MSS on such systems is NP-hard and that MSS is equivalent to exponential mean square stability (EMSS) and to stochastic stability (SS).**

I. INTRODUCTION

Wireless control networks (WCN) are distributed control systems where the communication between sensors, actuators, and computational units is supported by a wireless communication network. The use of WCN in industrial automation results in flexible architectures and generally reduces installation, debugging, diagnostic and maintenance costs with respect to wired networks (see e.g. [1], [2] and references therein). However modeling, analysis and design of (wireless) networked control systems (NCSs) are challenging open research problems since they require to take into account the joint dynamics of physical systems, communication protocols and network infrastructures. Recently, a huge effort has been made in scientific research on NCSs, see e.g. [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] and references therein for a general overview. In this domain it has been shown (e.g. in [14], [15], [16], [17]) that discrete-time Markov-jump linear systems (MJLS, [18]) represent a promising mathematical model to jointly take into account the dynamics of a physical plant and non-idealities of wireless communication such as packet losses. A MJLS is, basically, a switching linear system where the switching signal is a Markov chain. The transition probability matrix (TPM) of the Markov chain can be used to model the stochastic process that rules packet losses due to wireless communication. However, in most real cases, such probabilities cannot be computed exactly and are time-varying. We can take into account this aspect by assuming that the Markov chain of a MJLS is time-inhomogeneous, i.e. a Markov chain having its TPM varying over time, with variations that are arbitrary within a polytopic set of stochastic matrices. Given such mathematical model, the first problem to be addressed is the (mean square) stability problem. Some recent work addressed the above problem: in [19] a sufficient condition for stochastic stability in terms of linear matrix inequality feasibility problem is provided, while in [20] a sufficient condition for mean square stability (MSS) of system with interval transition probability matrix, which in turn can be represented as a convex polytope [21], is presented in relation to spectral radius; in general, only sufficient stability conditions have been derived for MJLS with time-inhomogeneous Markov chains having TPM arbitrarily varying within a polytopic set of stochastic matrices. As the *main contribution* of this paper, we provide necessary and sufficient conditions for MSS of discrete-time MJLS with time-inhomogeneous Markov chains. Such conditions require to decide whether the joint spectral radius (JSR) of a finite family of matrices is smaller than 1. While it is well known that the stability analysis problem for general switching systems (i.e. deciding whether the JSR is smaller than 1) is NP-hard [22], we prove that it is NP-hard even for the matrices structure deriving from our particular model. We also prove that MSS is equivalent to exponential mean square stability and to stochastic stability, and present an illustrative example showing that having the spectral radius smaller than one for each matrix Λ associated to the second moment of the state vector $x(k)$ is not enough to ensure the stability of the time-inhomogeneous system, and that perturbations on values of TPM can make a stable system unstable.

The *notation* used throughout is standard. The sets of all positive and nonnegative integers are represented by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The n-dimensional complex Euclidean space is indicated by $\mathbb{C}^n$, while a set of linear maps between two complex Euclidean spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ is denoted by $\mathbb{C}^{m\times n}$ and is encoded through a set of $m\times n$ complex matrices. The conjugate of a complex matrix M is denoted by $\bar{M}$, while the superscript * indicates the conjugate transpose of a matrix, and ^{T} indicates the transpose. Clearly for a set of real matrices, denoted by $\mathbb{R}^{m\times n}$, * and ^{T} have the same meaning. We indicate with $\mathbb{C}_*^{n\times n}$ the set of Hermitian matrices, and with $\mathbb{C}_+^{n\times n}$ the set of positive semi-definite matrices. The $n\times n$ identity matrix is denoted by $\mathbb{I}_n$. Unless otherwise stated, $\|\cdot\|$ will indicate the standard norm in $\mathbb{C}^n$, and, for $M\in\mathbb{C}^{m\times n}$, $\|M\|$ will denote the induced uniform norm in $\mathbb{C}^{m\times n}$. The linear space made up of all N sequences of complex matrices $K=[K_1,...,K_N]$, with $K_i\in\mathbb{C}^{m\times n}$, $i\in\mathbb{N}$, is indicated by $\mathbb{H}^{m,n}$. Finally, $\mathbf{E}[\cdot]$ stands for the mathematical expectation of the underlying scalar valued random variables.

II. DISCRETE-TIME TIME-INHOMOGENEOUS MARKOV JUMP LINEAR SYSTEMS

Let us consider a probability space $(\Omega,\mathcal{F},Pr)$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of events and Pr is

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the probability measure. Let $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ be a Markov chain defined on the probability space, which takes values in a finite set $\mathcal{N} \triangleq \{1, \dots, N\}$. For every $k \in \mathbb{N}_0$ let us define the transition probability as

$$p_{ij}(k) = \Pr\{\theta(k+1) = j \mid \theta(k) = i\} \geq 0, \; \sum_{j=1}^{N} p_{ij}(k) = 1. \quad (1)$$

The associated TPM $P(k)$ is a stochastic $N \times N$ matrix with entries $p_{ij}(k)$. In this work we assume that $P(k)$ is unknown and time-varying within a bounded set.

Assumption 1: TPM $P(k)$ is **polytopic**, i.e. $\forall k \in \mathbb{N}_0$

$$P(k) = \sum_{l=1}^{L} \lambda_l(k) P_l, \; \lambda_l(k) \geq 0, \; \sum_{l=1}^{L} \lambda_l(k) = 1, \quad (2)$$

where $\{P_l\}_{l \in \mathcal{L} \triangleq \{1,\dots,L\}} \triangleq \mathcal{P}_L$ is a given set of TPMs, which are the vertices of a convex polytope.

Remark 1: This assumption is not restrictive, since the polytopic uncertainty model is widely used for robust control of time-homogeneous MJLS (see e.g. [23]) and is considered to be more general than the partly known element model of TPM uncertainties; furthermore, also the interval TPM can be represented as a convex polytope [21].

The related noiseless autonomous discrete-time time-inhomogeneous MJLS $(\mathcal{S})$ is described by

$$\begin{cases} x(k+1) = A_{\theta(k)} x(k), \\ x(0) = x_0, \; \theta(0) = \theta_0 \end{cases} \quad (3)$$

where $x(k) \in \mathbb{C}^n$ is the state vector, $A_{\theta(k)} \in \mathbb{C}^{n \times n}$ is the state matrix associated with the operational mode of the system, $x(0)$ and $\theta(0)$ are initial conditions.

Remark 2: When studying MJLS, it is a standard practice to work with complex fields [18], but one can also work with real-valued ones, by considering complex matrices acting on $\mathbb{C}^{n \times n}$ as real operators acting on $\mathbb{R}^{2n \times 2n}$ [24].

The set $\mathcal{N}$ comprises the operational modes of the system $(\mathcal{S})$ and for each possible value of $\theta(k) = i, \; i \in \mathcal{N}$, we denote the state matrix associated with the i-th mode by $A_i = A_{\{\theta(k)=i\}}$. Thus, $A = [A_1, ..., A_N] \in \mathbb{H}^{n,n}$.

For a set $\Theta \in \mathcal{F}$, let us define the indicator function $\mathbf{1}_\Theta$ in the usual way [18, p. 31], that is, $\forall \omega \in \Omega$,

$$\mathbf{1}_\Theta(\omega) = \begin{cases} 1 & \text{if } \omega \in \Theta, \\ 0 & \text{otherwise.} \end{cases} \quad (4a)$$

Notice that, $\forall i \in \mathbb{N}$,

$$\mathbf{1}_{\{\theta(k)=i\}}(\omega) = 1 \text{ if } \theta(k)(\omega) = i, \text{ and } 0 \text{ otherwise.} \quad (4b)$$

Thus, we have that

$$\mathbf{E}[x(k)] = \sum_{i=1}^{N} \mathbf{E}[x(k) \mathbf{1}_{\{\theta(k)=i\}}],$$

$$\mathbf{E}[x(k)x^*(k)] = \sum_{i=1}^{N} \mathbf{E}[x(k)x^*(k) \mathbf{1}_{\{\theta(k)=i\}}].$$

Following the standard workflow for MJLS [18, p. 31], for $k \in \mathbb{N}_0$, let us use the following notation:

$$Q(k) \triangleq [Q_1(k), \dots, Q_N(k)] \in \mathbb{H}^{n+}, \quad (5a)$$

$$Q_i(k) \triangleq \mathbf{E}[x(k)x^*(k) \mathbf{1}_{\{\theta(k)=i\}}] \in \mathbb{C}_+^{n \times n}, \quad (5b)$$

$$q(k) \triangleq [q_1(k), \dots, q_N(k)]^T \in \mathbb{C}^{Nn}, \quad (5c)$$

$$q_i(k) \triangleq \mathbf{E}[x(k) \mathbf{1}_{\{\theta(k)=i\}}] \in \mathbb{C}^n, \quad (5d)$$

where $\mathbb{H}^{n+} \triangleq \{K \in \mathbb{H}^{n*}; \; K_i \in \mathbb{C}_+^{n \times n}, \; i \in \mathcal{N}\}$, and $\mathbb{H}^{n*} \triangleq \{K = [K_1, \dots, K_N] \in \mathbb{H}^{n,n}; \; K_i \in \mathbb{C}_*^{n \times n}, \; i \in \mathcal{N}\}$.

This permits us to define the expected value of $x(k)$ as

$$\mu(k) \triangleq \mathbf{E}[x(k)] = \sum_{i=1}^{N} q_i(k) \in \mathbb{C}^n, \quad (6)$$

and the second moment of $x(k)$ as

$$\mathcal{Q}(k) \triangleq \mathbf{E}[x(k)x^*(k)] = \sum_{i=1}^{N} Q_i(k) \in \mathbb{C}_+^{n \times n}. \quad (7)$$

Now, we are ready to recall the definition of the mean square stability [18, pp. 36–37].

Definition 1 (Mean square stability): We say that MJLS $(\mathcal{S})$ is mean square stable (MSS), if for every initial state $x_0 \in \mathbb{C}^n$ and for every initial probability distribution of $\theta(0)$

$$\lim_{k \to \infty} \mu(k) = 0 \text{ and } \lim_{k \to \infty} \mathcal{Q}(k) = 0. \quad (8)$$

We can easily see that the recursive equations for $q_i(k)$ and $Q_i(k)$ in the time-inhomogeneous case have the same structure as the time-homogeneous case with known probability matrix [18, p. 32], and the extension to this more general case is done in the following manner:

Proposition 1: Consider the system $\mathcal{S}$. $\forall k \in \mathbb{N}_0, \; j \in \mathcal{N}$

1) $q_j(k+1) = \sum_{i=1}^{N} p_{ij}(k) A_i q_i(k)$
2) $Q_j(k+1) = \sum_{i=1}^{N} p_{ij}(k) A_i Q_i(k) A_i^*$

Proof: Regarding the first statement, we have that

$$\begin{aligned} q_j(k+1) &= \mathbf{E}[A_{\theta(k)} x(k) \mathbf{1}_{\{\theta(k+1)=j\}}] \\ &= \sum_{i=1}^{N} \mathbf{E}[A_i x(k) \mathbf{1}_{\{\theta(k)=i\}} \mathbf{1}_{\{\theta(k+1)=j\}}] \\ &= \sum_{i=1}^{N} A_i \mathbf{E}[x(k) \mathbf{1}_{\{\theta(k)=i\}} \Pr\{\theta(k+1) = j \mid \theta(k) = i\}] \\ &= \sum_{i=1}^{N} A_i q_i(k) p_{ij}(k), \end{aligned}$$

which proves the first result. Similarly, for the next statement:

$$\begin{aligned} Q_j(k+1) &= \sum_{i=1}^{N} \mathbf{E}[A_i x(k) (A_i x(k))^* \mathbf{1}_{\{\theta(k)=i\}} \mathbf{1}_{\{\theta(k+1)=j\}}] \\ &= \sum_{i=1}^{N} A_i Q_i(k) A_i^* p_{ij}(k). \end{aligned}$$

■

Another useful result regards the inequality between $\|q(k)\|$ and $\|Q(k)\|_1$:

Proposition 2: Consider the system $\mathcal{S}$. $\forall k \in \mathbb{N}_0$

$$\|q(k)\|^2 \leq n \|Q(k)\|_1 \quad (9)$$

Proof: See [18, p. 35, the proof of Proposition 3.6]. ■

For $K = [K_1, \dots, K_N] \in \mathbb{H}^{m,n}$, let us denote the vectorization transformation $\varphi(K_i) \triangleq \text{vec}(K_i), \; i \in \mathcal{N}$ [25].

Indicating by $(K_i)_{\cdot j}$ the j-th column of K_i, we have

$$\operatorname{vec}(K_i) \triangleq \begin{bmatrix} (K_i)_{\cdot 1} \\ \vdots \\ (K_i)_{\cdot n} \end{bmatrix} \in \mathbb{C}^{nm},\ \hat{\varphi}(K) \triangleq \begin{bmatrix} \varphi(K_1) \\ \vdots \\ \varphi(K_N) \end{bmatrix} \in \mathbb{C}^{Nnm} \tag{10}$$

The spaces $\mathbb{H}^{m,n}$ and $\mathbb{C}^{Nnm}$ are uniformly homeomorphic [26, p. 117] through the linear mapping $\hat{\varphi}(\cdot)$ [18, p. 17].

Let us indicate by $\otimes$ a Kronecker product defined in the usual way [27]. For any X, Y, Z, M given matrices of appropriate size, the following properties are satisfied:

$$(X+Y)\otimes(Z+M) = X\otimes Z+Y\otimes Z+X\otimes M+Y\otimes M \tag{11a}$$

$$\varphi(XYZ) = (Z^T \otimes X)\varphi(Y) \tag{11b}$$

As for time-homogeneous case [18, pp. 33-35], also here, via application of (5b), Proposition 1, (10) and (11) to (5a), we have that

$$\hat{\varphi}(Q(k+1)) = \Lambda(k)\hat{\varphi}(Q(k)), \tag{12}$$

where $\Lambda(k) \in \mathbb{C}^{Nn^2\times Nn^2}$ is a matrix associated to the second moment of $x(t)$, defined in the following way:

$$\Lambda(k) \triangleq (P^T(k) \otimes \mathbb{I}_{n^2})diag[\bar{A}_i \otimes A_i], \tag{13}$$

in which we use the block diagonal matrix $diag[\bar{A}_i \otimes A_i] \triangleq$

$$\begin{bmatrix} \bar{A}_1 \otimes A_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 \otimes A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A_N} \otimes A_N \end{bmatrix} \tag{14}$$

Proposition 3: The matrix $\Lambda(k)$ is polytopic, i.e. $\forall k \in \mathbb{N}_0$

$$\Lambda(k) = \sum_{l=1}^{L} \lambda_l(k)\Lambda_l,\ \lambda_l(k) \geq 0,\ \sum_{l=1}^{L} \lambda_l(k) = 1, \tag{15a}$$

$$\Lambda_l \triangleq (P_l^T \otimes \mathbb{I}_{n^2})diag[\bar{A}_i \otimes A_i],\ P_l \in \mathcal{P}_L. \tag{15b}$$

Proof: Direct application of (2) and (11a) to (13). ∎

Let $\mathcal{A}_L$ be the set $\{\Lambda_l\}_{l\in\mathcal{L}}$ of all vertices of the polytope associated to the second moment of $x(k)$.

Remark 3: Recalling the definition of the convex hull [28, p. 14], we can write that $\forall k \in \mathbb{N}_0$, $\Lambda(k) \in \operatorname{conv}\mathcal{A}_L$.

From (12) we have that

$$\hat{\varphi}(Q(k)) = \Lambda(k-1)\Lambda(k-2)\cdots\Lambda(0)\hat{\varphi}(Q(0)). \tag{16}$$

It is well known, that the maximal rate of growth among all products of matrices from a finite set $\mathcal{M}$ is given by the joint spectral radius of $\mathcal{M}$, $\hat{\rho}(\mathcal{M})$, which is the generalization of spectral radius to sets of matrices.

III. Joint Spectral Radius

The notion of joint spectral radius (JSR) was introduced by Rota and Strang [29] and in the last decades it has been the subject of intense research due to its role in the study of wavelets, switching systems, approximation algorithms, curve design, and many other topics [24], [30]. In this section we present some results useful for our discussion.

Let $\mathcal{M}$ be a family of complex square matrices, that is $\mathcal{M} = \{M_l\}_{l\in\mathcal{L}}$, where $M_l \in \mathbb{C}^{n\times n}$. For each $k \in \mathbb{N}$, consider the set $\Pi_k(\mathcal{M})$ be a set of all possible products of length k whose factors are elements of $\mathcal{M}$, i.e.

$$\Pi_k(\mathcal{M}) = \{M_{l_k}M_{l_{k-1}}\cdots M_{l_1} \mid l_1,\ldots,l_k \in \mathcal{L}\}. \tag{17}$$

Definition 2 (Joint spectral radius): For any matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, consider the supremum among the normalized norms of all products in $\Pi_k(\mathcal{M})$, i.e.

$$\hat{\rho}_k(\mathcal{M}) \triangleq \sup_{\Pi\in\Pi_k(\mathcal{M})} \|\Pi\|^{1/k},\ k \in \mathbb{N}. \tag{18}$$

The joint spectral radius of $\mathcal{M}$ is defined as

$$\hat{\rho}(\mathcal{M}) = \lim_{k\to\infty} \hat{\rho}_k(\mathcal{M}). \tag{19}$$

The JSR of a bounded set of matrices has some interesting properties reported below.

Proposition 4 (Convex hull): The convex hull of a set has the same joint spectral radius as the original set, i.e.

$$\hat{\rho}(\operatorname{conv}\mathcal{M}) = \hat{\rho}(\mathcal{M}). \tag{20}$$

Proof: This result was first obtained by Barabanov [31]. See [32], [30] for further details. ∎

Proposition 5 (Convergence of matrix products): For any bounded set of matrices $\mathcal{M}$ and for any $k \in \mathbb{N}$, all matrix products $\Pi \in \Pi_k(\mathcal{M})$ converge to zero matrix as $k \to \infty$, if and only if $\hat{\rho}(\mathcal{M}) < 1$.

Proof: See the seminal work of Berger and Whang [33, Theorem I (b)]. ∎

These concepts are at the basis of our main results.

IV. Main Results

Theorem 1: The discrete-time Markov jump linear system $(\mathcal{S})$ with unknown and time-varying TPM $P(k) \in \operatorname{conv}\mathcal{P}_L$ is mean square stable if and only if $\hat{\rho}(\mathcal{A}_L) < 1$.

Proof: (Necessity: MSS $\Rightarrow \hat{\rho}(\mathcal{A}_L) < 1$) By hypothesis,

$$\forall\, \mathcal{Q}(0) = \mathbf{E}[x_0x_0^*],\ \lim_{k\to\infty} \mathcal{Q}(k) = 0.$$

From (7), we have that

$$\lim_{k\to\infty} \sum_{i=1}^{N} Q_i(k) = 0,\ Q_i(k) \in \mathbb{C}_+^{n\times n}.$$

Thus, from (5a) follows that

$$\lim_{k\to\infty} Q(k) = 0. \tag{21}$$

The mapping $\hat{\varphi}(\cdot)$ is uniform homeomorphic, so also

$$\lim_{k\to\infty} \hat{\varphi}(Q(k)) = 0.$$

Applying (16), we obtain

$$\lim_{k\to\infty} \Lambda(k-1)\Lambda(k-2)\cdots\Lambda(0)\hat{\varphi}(Q(0)) = 0.$$

From Proposition 5 and Remark 3, this last statement is true for every $Q(0)$ if and only if $\hat{\rho}(\operatorname{conv}\mathcal{A}_L) < 1$. From Proposition 4 follows the thesis.

Now, let us prove the sufficiency (MSS $\Leftarrow \hat{\rho}(\mathcal{A}_L) < 1$). The first part of the proof follows the inverse pattern of the proof of the necessity:

$$\hat{\rho}(\mathcal{A}_L) < 1 \Rightarrow \lim_{k\to\infty} \mathcal{Q}(k) = 0,\ \forall\, \mathcal{Q}(0) = \mathbf{E}[x_0x_0^*]$$

by Propositions 4-5, Remark 3, (16), uniform homeomorphism between spaces $\mathbb{H}^{m,n}$ and $\mathbb{C}^{Nnm}$ through the mapping $\hat{\varphi}(\cdot)$, (5a) and (7).

To complete the proof, we need to show that

$$\hat{\rho}(\mathcal{A}_L) < 1 \Rightarrow \lim_{k\to\infty} \mu(k) = 0,\ \forall\, \mu(0) = \mathbf{E}[x_0].$$

We already have (21) from the first part of the proof. Together with (9) from Proposition 2, this implies that

$$\lim_{k\to\infty} \|q(k)\| = \lim_{k\to\infty} \sum_{i=1}^{N} \|q_i(k)\| = 0.$$

This implies the thesis and concludes the proof. ∎

While it is well known that the stability analysis problem for general switching systems (i.e. deciding whether the JSR is smaller than 1) is NP-hard [22], we prove that it is NP-hard even for our particular model.

Theorem 2: Given a discrete-time Markov jump linear system $(\mathcal{S})$ with unknown and time-varying transition probability matrix $P(k) \in \operatorname{conv} \mathcal{P}_L$, unless $P = NP$, there is no polynomial-time algorithm that decides whether it is MSS.

Proof: Our proof works by reduction from the matrix semigroup stability, which is well known to be NP-hard [24, Theorem 2.4 and Theorem 2.6]. In this problem, one is given a set of two matrices $\mathcal{M} = \{M, M'\} \subset \mathbb{Q}_+^{r\times r}$ ($\mathbb{Q}_+$ is the set of nonnegative rational numbers), with $M = [m_{ij}]$ and $M' = [m'_{ij}]$, and one is asked whether the product of any sequence of matrices M, M' of length k converges to the zero matrix when $k \to \infty$. Let us consider a particular instance $\mathcal{M} = \{M, M'\} \subset \mathbb{Q}_+^{r\times r}$ of the matrix semigroup stability problem: we will build a discrete-time MJLS $\mathcal{S}$ with set of (scalar) state matrices $\{a_1, \ldots, a_N\} \subset \mathbb{R}_+, N = r + 1$, ($\mathbb{R}_+$ is the set of nonnegative real numbers), with unknown and time-varying TPMs $P(k) \in \operatorname{conv} \mathcal{P}_L, \mathcal{P}_L = \{P, P'\} \subset \mathbb{R}_+^{N\times N}$, where $P = [p_{ij}]$ and $P' = [p'_{ij}]$ are stochastic matrices, and prove that $(\mathcal{S})$ is MSS if and only if the set $\mathcal{M}$ is stable.

By (15b) it follows that

$$\mathcal{A} = P^T diag[a_i^2],\ \mathcal{A}' = (P')^T diag[a_i^2],\ i \in \mathcal{N}.$$

Our construction is as follows.

Assign arbitrarily for $j = 1, \ldots, r$

$$a_j^2 \in \mathbb{Q}_+,\ a_j^2 \geq r \max\{m_{1j}, \ldots, m_{rj}, m'_{1j}, \ldots, m'_{rj}\}.$$

Assign for $i, j = 1, \ldots, r$

$$p_{ij} \triangleq \frac{m_{ji}}{a_i^2} \in [0, \frac{1}{r}], \quad p'_{ij} \triangleq \frac{m'_{ji}}{a_i^2} \in [0, \frac{1}{r}].$$

Assign for $i = 1, \ldots, N$

$$p_{iN} \triangleq 1 - \sum_{j=1}^{r} p_{ij} \in [0, 1], \quad p'_{iN} \triangleq 1 - \sum_{j=1}^{r} p'_{ij} \in [0, 1].$$

Assign $a_N \triangleq 0$ and $p_{Nj} = p'_{Nj} \triangleq \frac{1}{N}$ for $j = 1, \ldots, N$. As a consequence of the above assignments, it follows that P, P' are stochastic matrices and that

$$\mathcal{A} = \begin{bmatrix} M & 0 \\ R & 0 \end{bmatrix},\ \mathcal{A}' = \begin{bmatrix} M' & 0 \\ R' & 0 \end{bmatrix},$$

with $R, R' \in \mathbb{Q}_+^r$. By Theorem 1 $\mathcal{S}$ is MSS if and only if the JSR of $\{\mathcal{A}, \mathcal{A}'\}$ is smaller than 1. From this, it is straightforward to see that $\mathcal{S}$ is MSS if and only if $\mathcal{M}$ is stable. This concludes the proof. ∎

Remark 4: It is not known (to the best of our knowledge) whether the matrix semigroup stability problem is Turing decidable (say, for matrices with rational entries). Thus, the above proof does not allow us to conclude that MSS is undecidable for MJLS with polytopic unknown and time-varying transition probability matrices. This is why we only claim that the stability problem is NP-hard.

The next theorem links MSS to EMSS and to SS.

Theorem 3 (Stability equivalence): The following assertions are equivalent.

(a) The system $(\mathcal{S})$ is mean square stable (MSS).

(b) The system $(\mathcal{S})$ is exponentially mean square stable (EMSS), i.e. for some $\beta \geq 1,\ 0 < \zeta < 1$, we have for all initial states $x_0 \in \mathbb{C}^n$ and all initial probability distributions of $\theta(0)$,

$$\mathbf{E}[\|x(k)\|^2] \leq \beta\zeta^k \|x_0\|_2^2,\ k \in \mathbb{N}_0. \tag{22}$$

(c) The system $(\mathcal{S})$ is stochastically stable (SS), i.e. for all $x_0 \in \mathbb{C}^n$ and all initial probability distributions of $\theta(0)$,

$$\sum_{k=0}^{\infty} \mathbf{E}[\|x(k)\|^2] < \infty. \tag{23}$$

Proof: It is trivially verified that $[(b) \Rightarrow (c)]$. Thus, let us show that $[(c) \Rightarrow (a)]$. First, let us note that

$$\mathbf{E}[\|x(k)\|^2] = \mathbf{E}[\operatorname{tr}(x(k)x^*(k))] = \operatorname{tr}(\mathcal{Q}(k)) \geq 0, \tag{24}$$

where $tr(\cdot)$ denotes the trace operator. Therefore, from (23)

$$\lim_{k\to\infty} \operatorname{tr}(\mathcal{Q}(k)) = 0.$$

Accordingly, as stated in [18, p. 44, within the proof of Proposition 3.24], this implies that

$$\lim_{k\to\infty} \mathcal{Q}(k) = 0,\ \forall x_0,\ \forall \theta_0.$$

We have already seen in the proof of the sufficiency of Theorem 1 that this implies MSS of the system $(\mathcal{S})$. Hence, this part of the proof is concluded.

Moreover, as a result we have also that $[(b) \Rightarrow (a)]$.

Now, let us show that $[(a) \Rightarrow (b)]$. From Theorem 1 we know that if the system $(\mathcal{S})$ is MSS, then $\hat{\rho}(\mathcal{A}_L) < 1$. Since

$$\lim_{k\to\infty} \|\Lambda(k)\cdots\Lambda(0)\|^{\frac{1}{k}} = \hat{\rho}(\mathcal{A}_L),$$

by the radical test for infinite series, we can state that

$$\|\Lambda(k-1)\cdots\Lambda(0)\| < \zeta^k,\ \forall k \geq k',\ \forall \zeta \in (\hat{\rho}(\mathcal{A}_L), 1),$$

for some integer $k' \geq 0$. With

$$\beta' = \zeta^{-k'} \sup_{\Pi \in \Pi_j[\mathcal{A}_L],\ 0 \leq j \leq k'} \|\Pi\|, \quad \beta' \geq 1,$$

we obtain that

$$\|\Lambda(k-1)\cdots\Lambda(0)\| \leq \beta'\zeta^k,\ \forall k \in \mathbb{N}_0. \tag{25}$$

Now, from (24), we have that

$$\begin{aligned} \mathbf{E}[\|x(k)\|^2] &= \sum_{i=1}^{N} \operatorname{tr}(\mathbf{E}[x(k)x^*(k)\mathbf{1}_{\{\theta(k)=i\}}]) \\ &= \sum_{i=1}^{N} \operatorname{tr}(Q_i(k)) \leq n \sum_{i=1}^{N} \|Q_i(k)\|. \end{aligned}$$

Since we are working on a finite-dimensional linear space, any two norms are equivalent [34, Theorem 4.27], and we can make use of Frobenius norm defined for $M \in \mathbb{C}^{m,n}$ as

$$\|M_i\|_F \triangleq \sqrt{\operatorname{tr}(M_i^* M_i)} = \|\operatorname{vec}(M_i)\|_F = \|\varphi(M_i)\|_F \quad (26)$$

to obtain $\|Q_i(k)\|_F \leq \|\Lambda(k-1)\cdots\Lambda(0)\|_F \ \|\hat{\varphi}(Q(0))\|_F$, which holds $\forall i \in \mathcal{N}$. Considering that for Frobenius norm $\|\hat{\varphi}(Q(0))\|_F = \|Q(0)\|_1$, we have that

$$\mathbf{E}[\|x(k)\|^2] \leq nN \, \|\Lambda(k-1)\cdots\Lambda(0)\|_F \ \|Q(0)\|_1.$$

Since (25) holds for any equivalent norm, and having

$$\|Q(0)\|_1 = \sum_{i=1}^{N} \|Q_i(0)\| \leq \sum_{i=1}^{N} \mathbf{E}[\|x_0\|^2 \mathbf{1}_{\{\theta(0)=i\}}] = \|x_0\|_2^2,$$

we can finally write

$$\mathbf{E}[\|x(k)\|^2] \leq nN\beta'\zeta^k \|x_0\|_2^2 = \beta\zeta^k \|x_0\|_2^2. \quad (27)$$

Thus, we have proved that $[(a) \Rightarrow (b)]$. All the remaining implications follow from the already proved ones. This concludes the proof. ■

V. ILLUSTRATIVE EXAMPLE

In order to show that having the spectral radius smaller than one for each matrix Λ associated to the second moment of $x(k)$ is not enough to ensure the (mean square) stability of the time-inhomogeneous system, let us consider the MJLS $\mathcal{S}$ with $N = 3$ operational modes, where the state matrices associated with the operational modes are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.48 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.3 & 0.13 \\ 0.16 & 1.14 \end{bmatrix}$$

and the time-varying probability matrix $P(k)$ is uncertain and belongs to a polytope with $L = 2$ vertices

$$P_1 = \begin{bmatrix} 0 & 0.35 & 0.65 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}.$$

Any probability matrix within a polytope is represented by

$$P(k) = \lambda(k)P_1 + (1-\lambda(k))P_2,\ 0 \leq \lambda(k) \leq 1.$$

Let us consider, for instance, also the matrix

$$P' = 0.5\ P_1 + 0.5\ P_2.$$

The spectral radii ρ of the matrices Λ are:

$$\rho(\Lambda_1) = 0.901601,\ \rho(\Lambda_2) = 0.905686,\ \rho(\Lambda') = 0.937965.$$

Thus, the time-homogeneous MJLS with TPM P_1, P_2 and P' are mean square stable [18].

However, the time-inhomogeneous system having this TPMs is not (mean square) stable, because the joint spectral radius, calculated with the JSR toolbox [35], is

$$\hat{\rho}(\mathcal{A}_L) = [\hat{\rho}_{min}(\mathcal{A}_L), \hat{\rho}_{max}(\mathcal{A}_L)] = [1.024442, 1.031096]$$

This shows us that perturbations on transition probability matrix P can make a stable MJLS system unstable.

In order to present this result visually, we report one possible dynamical behavior of the system. For $x_0 = [100; 85]$ and the initial probability distribution $p_0 = [0.33, 0.34, 0.33]$, we have obtained the following system trajectories.

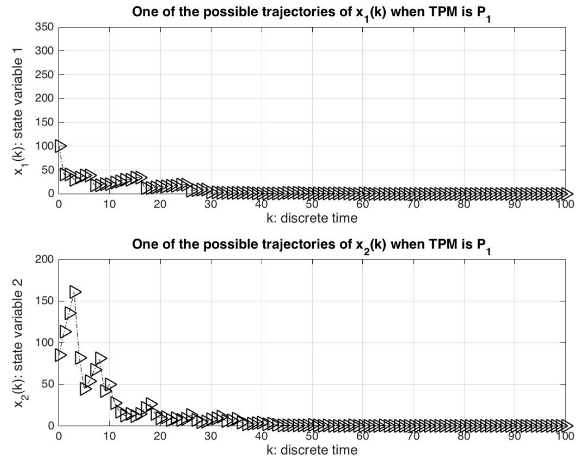

Fig. 1. One of the possible trajectories of $x(k)$ when TPM is P_1

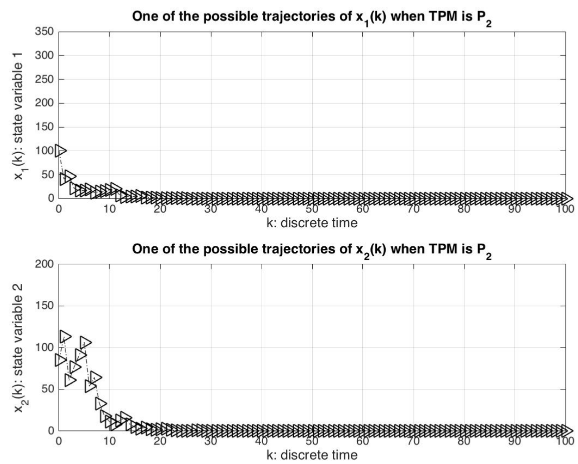

Fig. 2. One of the possible trajectories of $x(k)$ when TPM is P_2

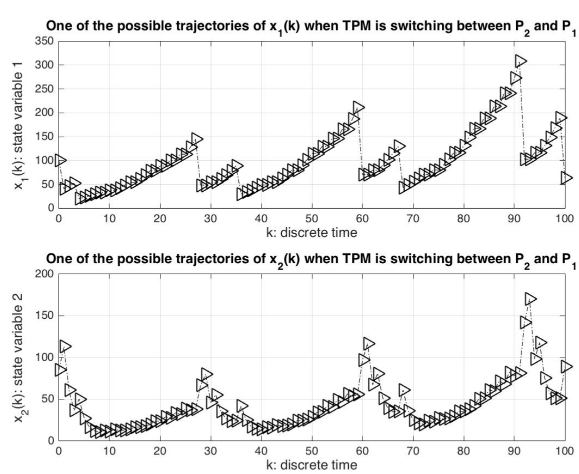

Fig. 3. Trajectory of $x(k)$ when TPM is switching between P_1 and P_2

Figure 1 shows us a trajectory of the system state vector having only the time-homogeneous transition probability matrix P_1, while Figure 2 presents a system trajectory when TPM used is always P_2. Finally, Figure 3 reveals a trajectory of the system state vector when the transition probability matrix is time-inhomogeneous and is switching between P_1 and P_2, evincing instability of the system.

VI. CONCLUSIONS

The discrete-time time-inhomogeneous Markov jump linear systems with polytopic uncertainties on transition probabilities present a promising mathematical model to jointly take into account the dynamics of a physical plant and non-idealities of wireless communication such as packet losses. It is well suited for modeling of the control systems operating on the fading communication channels, which have probabilistic and time-varying behavior. We plan to apply this model to such scenarios, in order to study first the cases with bounded disturbances, and then the problems of linear quadratic regulation, H_∞ control, fault and intrusion detection among others.

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