

On Internal Stability of Wirelessly Controlled Stochastic Linear Systems With Channel State Detection and Message Dropout Compensation*

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Abstract—This paper develops a hidden Markov jump linear system model to analyze the internal stability of stochastic linear systems with wireless state-feedback control under imperfect channel-state information. It considers the current and future control inputs that actuators can apply if future control messages are lost. The dropout compensation strategy supports scaling inputs to actuators when needed. We present coupled Lyapunov-like equations for the system’s internal stochastic stability and show that linear matrix inequality (LMI) formulations can be derived to verify stochastic stabilizability and to design a stabilizing feedback control. We validate the results with an illustrative numerical example that demonstrates the practicality and effectiveness of our approach.

Index Terms—Control over communications; Markov processes; stability of hybrid systems.

I. INTRODUCTION

Wireless networked control systems (WNCSs), with their critical roles in applications like intelligent transportation, industrial automation, smart grids, and telesurgery, represent a significant area of interest for both industry and academia, as noted in [1], [2], and [3]. Research on estimation and control over unreliable links is a key focus within WNCSs, as discussed in [1], [3], [4], and references therein.

An important finding from [4], [5] and [6] is that accurate and timely knowledge of communication link conditions can significantly enhance closed-loop performance. However, obtaining perfect wireless channel state information (CSI) may be difficult in real-world communication scenarios, so only imperfect or statistical CSI might be available, as discussed in [7], [8], and [9]. Model predictive control (MPC)-based algorithms in [3] depend on transmitting a sequence of future control inputs that actuators can use if upcoming messages are lost. This method increases processing and transmission delays, the likelihood of message loss, and network traffic due to larger message sizes. The number of future control inputs to send becomes a key design decision, balancing the disadvantages of larger messages with the benefits of improved control. [6] introduced a framework to analyze this trade-off under imperfect CSI and proposed a hybrid message dropout compensation scheme that combined transmitting multiple control inputs with scaling inputs to actuators when needed. It models CSI uncertainties using the detector-based

approach from [10] and [11], extending it to the one-time-step-delayed channel-state-observation setting of [4] and [5], where wireless link dynamics are represented by finite-state Markov channels (FSMCs, as in [12]). [6] presented a practical hidden Markov jump linear system (HMJLS) model, which is especially useful for solving optimal control problems. [13] extended it to the output-feedback setting. For the stability analysis, they followed the idea in [4] to build an equivalent model based on the most complete and updated information available to the actuators. This paper shows that the stability analysis needs only part of this information, significantly reducing the augmented system’s state space. The proposed model maintains accuracy while simplifying the analysis, enabling us to formulate coupled Lyapunov-like equations for the system’s internal stochastic stability and to develop LMI formulations for verifying stochastic stabilizability and designing a stabilizing feedback control.

This paper’s *contribution* is twofold.

- We propose an enhanced HMJLS *model* for discrete-time linear stochastic systems with hybrid message dropout compensation and imperfect CSI, especially suited for examining *stabilizability* and *internal stability*.
- We also develop *LMI* formulations to verify stochastic *stabilizability* and *design stabilizing feedback* controls; we validate these with a numerical example.

The main *novelty* lies in the HMJLS model, which departs from the tradition established in [4], [6], [13] of using all the information available to the actuators and instead identifies a reduced information set required for accurate stability and stabilizability analysis.

This paper is *organized* as follows. Section II introduces the FSMC model of wireless links, Section III describes the closed-loop system model, while Section IV presents the first main result—an improved equivalent model of the closed-loop system that simplifies and enables the internal stability analysis discussed in Section V. Section VI verifies the results through a numerical example, and Section VII concludes the paper.

Notation: $\mathbb{R}$, $\mathbb{Z}^+$, and $\mathbb{Z}^{0+}$ denote the sets of real numbers, positive integers, and nonnegative integers. $\mathbb{R}^n$ represents n -dimensional Euclidean space. Symmetric positive definite and positive semidefinite matrices Z are indicated by $Z \succ 0$ and $Z \succeq 0$. $Z^\top$ denotes the transpose of matrix Z . $\rho(Z)$ is the spectral radius of a square matrix Z . $\mathbb{P}$ denotes the probability of an event, and $\mathbb{E}$ signifies the expected value of a random variable. I_N indicates the identity matrix of size N . Throughout this paper, N is the number of FSMC states, and M is the number of detector outputs, with $M \leq N$.

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Additionally, e_i represents the column vector of the standard basis of $\mathbb{R}^N$, where all components are zero except for the i th, which is one. The column vectors of the standard basis of $\mathbb{R}^M$ are denoted by $\hat{e}_\mu$, with $\mu \in \mathbb{Z}^+$ and $\mu \leq M$. Moreover, $\oplus$ indicates the direct sum, resulting in a diagonal matrix with the addends on the main diagonal. $\otimes$ represents the Kronecker product. Lastly, $\mathbf{1}$ is a vector of appropriate size with all elements equal to one, and $\star$ indicates transposed blocks in a symmetric matrix, i.e., $\begin{bmatrix} Z_1 & Z_2 \\ \star & Z_3 \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_2^\top & Z_3 \end{bmatrix}$.

II. WIRELESS LINK MODEL

In a WNCS environment, controllers communicate with actuators and sensors through wireless links, whose time-varying behavior can be accurately modeled using FSMC models [14]. An FSMC consists of a finite set of states, $\mathcal{S} \triangleq \{s_i\}_{i=1}^N$, each with a specific message dropout probability. A discrete-time Markov chain $\{\theta_k\}$ determines the active channel state. Formally, $\forall k \in \mathbb{Z}^{0+}$,

$$\mathbb{P}(\theta_k = s_j \mid \theta_{k-1} = s_i) = p_{ij} \geq 0, \quad \sum_{j=1}^N p_{ij} = 1, \quad (1)$$

$$\mathbb{P}(\delta_k = 1 \mid \theta_k = s_j) = \hat{\delta}_j, \quad \mathbb{P}(\delta_k = 0 \mid \theta_k = s_j) = 1 - \hat{\delta}_j, \quad (2)$$

where $\{\delta_k\}$ is a binary stochastic process used to model message loss that occurs between the transmitter and receiver.

For clarity, we organize the channel-state transition probabilities (TPs) in the transition probability matrix (TPM)

$$P_c \triangleq [p_{ij}]_{i,j=1}^N, \quad (3a)$$

and the success (or, conversely, failure) probabilities of delivering a message at time k into the following matrices:

$$P_s \triangleq [p_{ij}\hat{\delta}_j]_{i,j=1}^N, \quad P_f = P_c - P_s. \quad (3b)$$

We also denote the channel state probability at time k as

$$\mathbb{P}(\theta_k = s_i) = \pi_i(k). \quad (4)$$

Wireless receivers perform CSI measurements that are sent back to the transmitters. In sensing links, the controller is the receiver with direct access to the measured CSI. In actuation links, the controller is the transmitter. Usually, it can access the measured CSI through a reporting mechanism similar to transmission acknowledgment, which introduces a delay. This delay makes the optimal state-feedback control problem the main challenge in unconstrained output-feedback control over FSMC links [5]. Therefore, this paper focuses on the internal stability (as formally defined in Section V) of the wirelessly controlled stochastic discrete-time linear system and investigates the impact of actuation link CSI measurement noise, including cases with partial or no observation of FSMC states. Internal stability is essential in the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control synthesis problems for a broad class of Markov jump linear systems (MJLSs) that use a detector to observe their operational modes, as detailed, for instance, in [10] and [15]. We adopt the detector-based approach as follows. Refer to $\hat{\theta}_k \in \{\hat{s}_\mu\}_{\mu=1}^M$ as the measured CSI, which is obtained from a detector that uses an emission probability matrix (EPM)

$$P_e \triangleq [\alpha_{i\mu}]_{i,\mu=1}^{N,M}, \quad (5a)$$

$$\mathbb{P}(\hat{\theta}_k = \hat{s}_\mu \mid \theta_k = s_i) = \alpha_{i\mu} \geq 0, \quad \sum_{\mu=1}^M \alpha_{i\mu} = 1. \quad (5b)$$

Note that $M \leq N$ includes the following specific cases:

- *Perfect CSI*: $M = N$ and $P_e = I_N$ so that θ_k is known.
- *No CSI*: $M = 1$ and $P_e = \mathbf{1}$ so that θ_k is unknown.

Additionally, scenarios of partial observation using clusters are effectively modeled by a detector with $M < N$; see [10].

Remark 1: In an FSMC, briefly described by (1)–(5), for all $t \in \mathbb{Z}^{0+}$, $\hat{\theta}_t$ and δ_t are independent of each other and each depends only on θ_t . Additionally, for all $t \in \mathbb{Z}^+$, θ_t depends only on θ_{t-1} .

For simplicity in notation, define the diagonal matrices

$$P_\pi(k) = \bigoplus_{i=1}^N \pi_i(k), \quad P_\alpha(\mu) = \bigoplus_{i=1}^N \alpha_{i\mu}. \quad (6)$$

These matrices are crucial for effectively displaying the transition probabilities between the discrete states of the closed-loop system, as detailed in Section IV.

III. SYSTEM MODEL

In this section, consider the following controlled discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \quad (7)$$

where, for each $k \in \mathbb{Z}^{0+}$, $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$ are the system state and control input to actuators, A and B are the state and input matrices of appropriate size.

The *controller* uses a wireless link to send a control message $\hat{u}_k$ that includes the current input and $n_f \in \mathbb{Z}^{0+}$ tentative future inputs to account for potential control message dropouts after the current transmission:

$$\hat{u}_k = ([\bar{u}_{(k+t|k)}]_{t=0}^{n_f})^\top = [\bar{u}_{(k|k)}, \dots, \bar{u}_{(k+n_f|k)}]^\top. \quad (8)$$

The controller can obtain the CSI measured by the actuator's receiver but not the actual state of the FSMC. Additionally, as a transmitter, the controller does not measure CSI and cannot know the outcome of a control message transmission before sending it. This information is received from the actuator with a time-step delay. Consequently, all control inputs in the message $\hat{u}_k$ depend on the last available CSI measurement, which is $\hat{\theta}_{k-1}$. This paper analyzes the steady-state characteristics of the closed-loop system under state-feedback control of the form

$$\bar{u}_{(k+t|k)} = \bar{K}_{(t,\hat{\theta}_{k-1})} x_k, \quad (9)$$

where $t \in \mathbb{Z}^{0+}$ and $t \leq n_f$. From (8) and (9),

$$\hat{u}_k = \hat{K}_{(\hat{\theta}_{k-1})} x_k, \quad \hat{K}_{(\hat{\theta}_{k-1})}^\top = [\bar{K}_{(t,\hat{\theta}_{k-1})}^\top]_{t=0}^{n_f}. \quad (10)$$

For simplicity in notation, we set

$$K \triangleq \{K_\mu : K_\mu = \hat{K}_{(s_\mu)}\}_{\mu=1}^M. \quad (11)$$

In other words, K represents a set of infinite-horizon state-feedback controller gains, as in (10), associated with potential CSI measurements from the wireless channel state detector. Note that the controller's information set used to determine the control message in (8) is

$$\hat{\mathcal{I}}_k = \{(x_t)_{t=0}^k, (\delta_{t-1})_{t=1}^k, (\hat{\theta}_{t-1})_{t=1}^k\}. \quad (12)$$

Remark 2: As detailed in [6], based on the FSMC properties described in Remark 1, the control law in (9) does not depend on $(\delta_{t-1})_{t=1}^k$.

As receivers, actuators are aware of the controller's transmission result. Their information set used to determine the control input in (7) is

$$\mathcal{I}_k = \{ (\delta_t \hat{u}_t)_{t=0}^k, (\delta_t)_{t=0}^k \}. \quad (13)$$

If the control message dropout interval exceeds the number of tentative future control inputs n_f , the actuators apply the generalized control *message dropout compensation* to the last available control input. Specifically, this input is multiplied by a matrix $\Phi = \bigoplus_{i=1}^{n_u} \phi_i$, where $0 \leq \phi_i \leq 1$. Then, the control inputs to the actuators are provided as follows.

$$u_k = F \tilde{u}_k, \quad F \triangleq \tilde{e}_1^\top \otimes I_{n_u}, \quad (14)$$

$$\tilde{u}_k = \delta_k \hat{u}_k + (1 - \delta_k) H \tilde{u}_{k-1}, \quad \tilde{u}_{-1} = \mathbf{0}, \quad (15)$$

$$H \triangleq \sum_{i=1}^{n_f} \tilde{e}_i \otimes \tilde{e}_{i+1}^\top \otimes I_{n_u} + \tilde{e}_{n_f+1} \otimes \tilde{e}_{n_f+1}^\top \otimes \Phi, \quad (16)$$

where $\tilde{e}_i$ represents the column vector of the standard basis in $\mathbb{R}^{n_f+1}$. In (14), u_k extracts the first n_u elements from the buffer $\tilde{u}_k$, as described in (15). If a control message is successfully received, $\delta_k = 1$ and the message is stored in the buffer. If not, the buffer undergoes a linear transformation using matrix H : it removes the first n_u elements, shifts the remaining elements up by n_u positions, multiplies the last n_u elements by Φ , and places the result at the bottom of the buffer. See [6] and [13] for additional details on the advantages and practicality of this compensation mechanism and the closed-loop system architecture.

Assumption 1: The finite-state Markov chain $\{\theta_k\}$ is ergodic. Its steady-state probability distribution is $\pi = [\pi_i]_{i=1}^N$, where $\pi_i = \lim_{k \rightarrow \infty} \pi_i(k)$.

The next section integrates (1)–(16) to demonstrate that the controlled discrete-time linear system is a specific type of hidden Markov jump linear system.

IV. HIDDEN MARKOV JUMP LINEAR SYSTEM MODEL

The trajectories of the system (7)–(16) depend on realizations of the stochastic process $\{\delta_k\}$, which evolves according to (1)–(5). As discussed in [4], analyzing the stability of this type of system requires considering each control message transmission outcome from the actuators' perspective.

As in [4] and [6], we model the control-message dropout intervals using an auxiliary stochastic variable Δ_k :

$$\Delta_k = (1 - \delta_{k-1})(\Delta_{k-1} + 1). \quad (17)$$

It counts the consecutive control-message dropouts that occur before transmission at discrete time k . From (17), we have

$$\Delta_k = \ell \Leftrightarrow \delta_{k-1-\ell} = 1 \wedge \delta_{k-t} = 0 \quad \forall t \in \mathbb{Z}^+ : t \leq \ell. \quad (18)$$

Notably, if $\ell = 0$, then $\{t\}_{t=1}^0 = \emptyset$, that is, $\Delta_k = 0 \Leftrightarrow \delta_{k-1} = 1$.

In the following, $\mathcal{T}$ denotes the set of time points at which actuators successfully receive control messages. Formally,

$$\mathcal{T} \triangleq \{k : \delta_k = 1\}_{k \in \mathbb{Z}^{0+}} = \{\tau_{(m)}\}_{m \in \mathbb{Z}^{0+}}. \quad (19a)$$

From (17), (18), and (19a), for all $m \in \mathbb{Z}^{0+}$,

$$\tau_{(m)} \in \mathcal{T} \Rightarrow \delta_{\tau_{(m)}} = 1 \Rightarrow \Delta_{\tau_{(m)}+1} = 0, \quad (19b)$$

$$\tau_{(m+1)} = \tau_{(m)} + 1 + \Delta_{\tau_{(m)}+1}. \quad (19c)$$

Remarkably, (19b) shows that $\Delta_{\tau_{(m)}+1}$ can be considered part of the actuators' information set $\mathcal{I}_{\tau_{(m)}}$, while $\Delta_{\tau_{(m+1)}}$ is unknown beforehand.

To develop an HMJLS model for stability analysis, we define the following matrices. For any $h \in \mathbb{Z}^{0+}$,

$$\mathcal{A}_h \triangleq A^{h+1}, \quad \mathcal{B}_h \triangleq \sum_{i=0}^h A^{h-i} B F H^i. \quad (20)$$

The following proposition serves as the foundation for constructing the augmented state in the Markovian framework. It is a direct consequence of [6, Prop. 1] that covers the case of additive white Gaussian process noise.

Proposition 1: For any given initial system state x_0 , the control law that satisfies (10), and a realization of $(\delta_t)_{t=0}^{\tau_{(m)}+\ell}$, the system (7)–(16) is trace-equivalent to (17)–(21), meaning that the states $x_{\tau_{(m)}}$ and $x_{\tau_{(m)}+1+h}$ obtained from (7) and (21) will be the same for all $h \in \mathbb{Z}^{0+}$ such that $h \leq \ell$.

$$\begin{cases} \Delta_{\tau_{(m+1)}} = \ell \in \mathbb{Z}^{0+} \Rightarrow \forall h \in \mathbb{Z}^{0+} : h \leq \ell, \\ x_{\tau_{(m)}+1+h} = (\mathcal{A}_h + \mathcal{B}_h \hat{K}_{(\hat{\theta}_{\tau_{(m)}-1})}) x_{\tau_{(m)}}. \end{cases} \quad (21)$$

Proof: By construction, (17)–(21) describe the dynamics of (7)–(16) for the same initial system state x_0 , control law that satisfies (10), and realization of $(\delta_t)_{t=0}^{\tau_{(m)}+\ell}$. ■

To derive the stochastic description of (17)–(21) based on (1)–(6), consider the following expanded discrete states.

$$\varphi_{\tau_{(m+t)}} \triangleq (\Delta_{\tau_{(m+t)}}, \hat{\theta}_{\tau_{(m+t)} - \Delta_{\tau_{(m+t)}} - 2}), \quad \forall t \in \mathbb{Z}^+. \quad (22)$$

Remark 3: [6] uses $(\hat{\theta}_{\tau_{(m+t)}}, \Delta_{\tau_{(m+t)}}, \hat{\theta}_{\tau_{(m+t)} - \Delta_{\tau_{(m+t)}} - 2})$ to fully account for the information available to the actuators. This yields an HMJLS model with a significantly larger state space, as shown in Section VI.

From (19c) and (22), for any $t \in \mathbb{Z}^+$ and $\tau_{(m+t)} \in \mathcal{T}$,

$$\varphi_{\tau_{(m+t)}} = (\Delta_{\tau_{(m+t)}}, \hat{\theta}_{\tau_{(m+t-1)}-1}). \quad (23)$$

Consider ℓ and n in $\mathbb{Z}^{0+}$, μ and ν in $\mathbb{Z}^+$, $\mu \leq M$, $\nu \leq M$, $\Delta_{\tau_{(m+1)}} = \ell$, $\Delta_{\tau_{(m+2)}} = n$, $\hat{\theta}_{\tau_{(m)}-1} = \hat{s}_\mu$, and $\hat{\theta}_{\tau_{(m+1)}-1} = \hat{s}_\nu$, so that $\varphi_{\tau_{(m+1)}} = (\ell, \hat{s}_\mu)$ and $\varphi_{\tau_{(m+2)}} = (n, \hat{s}_\nu)$. From (21),

$$x_{\tau_{(m+1)}} = (\mathcal{A}_\ell + \mathcal{B}_\ell K_\mu) x_{\tau_{(m)}} \quad \text{if } \varphi_{\tau_{(m+1)}} = (\ell, \hat{s}_\mu), \quad (24)$$

$$x_{\tau_{(m+2)}} = (\mathcal{A}_n + \mathcal{B}_n K_\nu) x_{\tau_{(m+1)}} \quad \text{if } \varphi_{\tau_{(m+2)}} = (n, \hat{s}_\nu). \quad (25)$$

Consider $\tau_{(m+1)} \in \mathcal{T}$. By (10), (13), and (17)–(19), $x_{\tau_{(m+1)}}$, $\varphi_{\tau_{(m+1)}}$, and $x_{\tau_{(m)}}$ in (24) are part of the information set $\mathcal{I}_{\tau_{(m+1)}}$. In contrast, $x_{\tau_{(m+2)}}$ and $\varphi_{\tau_{(m+2)}}$ are unknown to actuators at time $\tau_{(m+1)}$. Therefore, we use (24) and (25) to identify the current and next HMJLS states at $\tau_{(m+1)}$:

- Current augmented state: $(x_{\tau_{(m)}}; \varphi_{\tau_{(m+1)}})$;
- Subsequent augmented state: $(x_{\tau_{(m+1)}}; \varphi_{\tau_{(m+2)}})$.

Remark 4: By examining all possible values of $\Delta_{\tau_{(m+2)}}$ in $\varphi_{\tau_{(m+2)}}$, we capture both successful and failed transmissions.

See [13, Sec. 6] for insights into representing the system at an arbitrary discrete time point.

For notational simplicity, we define the controlled system state matrix as $\Psi_{\varphi_{\tau(m+2)}} \triangleq \mathcal{A}_{\Delta_{\tau(m+2)}} + \mathcal{B}_{\Delta_{\tau(m+2)}} \hat{K}_{(\hat{\theta}_{\tau(m+1)} - 1)}$, so that (25) becomes

$$x_{\tau(m+2)} = \Psi_{\varphi_{\tau(m+2)}} x_{\tau(m+1)}. \quad (26)$$

Remark 5: The controlled system state matrix depends on the next, not the current, operational mode $\varphi_{\tau(m+2)}$. This difference renders the existing results on internal stabilizability, such as those in [10] and [15], inapplicable. The results in [6] remain valid but rely on a different, significantly larger state space. This paper's formulation greatly reduces the computational complexity of stability verification for any given set of control laws, as in (10), and, more importantly, enables LMI-based stabilizability tests suitable for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control synthesis problems.

In the following, we will use $\Psi_{(n,\nu)}$ to denote $\mathcal{A}_n + \mathcal{B}_n K_\nu$ whenever $\varphi_{\tau(m+2)} = (n, \hat{s}_\nu)$. Note from Remark 1 that n and $\hat{s}_\nu$ are independent, but both depend on the respective hidden states of the Markov chain $\{\theta_k\}$.

Theorem 1: The transition probabilities between the states defined by (23) for $t \in \{1, 2\}$ is

$$\begin{aligned} & \mathbb{P}(\varphi_{\tau(m+2)} = (n, \hat{s}_\nu) \mid \varphi_{\tau(m+1)} = (\ell, \hat{s}_\mu)) \\ &= \frac{\hat{e}_\mu^\top P_e^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_\alpha(\nu) P_s P_f^n P_s \mathbf{1}}{\hat{e}_\mu^\top P_e^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_s \mathbf{1}}. \end{aligned} \quad (27)$$

Proof: It is similar to the proof of [6, Th. 1]. It relies on (19a), (23), the chain rule of probability, the definition of conditional probability, the fact that an intersection of a set with itself is the set itself, the law of total probability, the commutative property of set intersection, Remark 1, and (1)–(6). ■

Corollary 1.1: In the perfect CSI case, (27) becomes (28):

$$\begin{aligned} & \mathbb{P}(\varphi_{\tau(m+2)} = (n, \hat{s}_\nu = s_j) \mid \varphi_{\tau(m+1)} = (\ell, \hat{s}_\mu = s_i)) \\ &= \frac{\pi_i(\tau(m) - 1) e_i^\top P_s P_f^\ell e_j e_j^\top P_s e_{i_1} e_{i_1}^\top P_f^n P_s \mathbf{1}}{\pi_i(\tau(m) - 1) e_i^\top P_s P_f^\ell P_s \mathbf{1}} \\ &= \frac{e_i^\top P_s P_f^\ell e_j e_j^\top P_s P_f^n P_s \mathbf{1}}{e_i^\top P_s P_f^\ell P_s \mathbf{1}}. \end{aligned} \quad (28)$$

Proof: Under perfect CSI, $M = N$ and $P_e = I_N$ so that $\hat{e}_\mu^\top P_e^\top = e_i^\top I_N = e_i^\top$, $e_i^\top P_\pi(\tau(m) - 1) = e_i^\top \pi_i(\tau(m) - 1) = \pi_i(\tau(m) - 1) e_i^\top$, and $P_\alpha(\nu = j) = e_j e_j^\top$. ■

Corollary 1.2: In the no CSI case, (27) is transformed into

$$\begin{aligned} & \mathbb{P}(\varphi_{\tau(m+2)} = (n, -) \mid \varphi_{\tau(m+1)} = (\ell, -)) \\ &= \frac{\mathbf{1}^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_s P_f^n P_s \mathbf{1}}{\mathbf{1}^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_s \mathbf{1}}, \end{aligned} \quad (29)$$

where “−” indicates that the CSI is unavailable.

Proof: Without CSI, $M = 1$, $P_e = \mathbf{1}$, $\hat{e}_\mu = \mathbf{1}$, $\hat{e}_\nu = \mathbf{1}$, and $P_\alpha(\nu) = I_N$. ■

Remark 6: The transition probabilities in (27) and (29) are time-inhomogeneous, meaning they change over time based

on the channel state probability distribution in $\tau(m) - 1$. Thus, stability analysis of (26) under (27) and (29) depends on Assumption 1 to ensure that the associated TPM converges to the corresponding time-homogeneous one. In the perfect CSI case in Corollary 1.1, Assumption 1 is not necessary.

For simplicity, we will abbreviate (27) as $q_{(\tau(m)-1, \ell, \mu, n, \nu)}$:

$$q_{(\tau(m)-1, \ell, \mu, n, \nu)} \triangleq \frac{\hat{e}_\mu^\top P_e^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_\alpha(\nu) P_s P_f^n P_s \mathbf{1}}{\hat{e}_\mu^\top P_e^\top P_\pi(\tau(m) - 1) P_s P_f^\ell P_s \mathbf{1}}.$$

Notice from (1)–(6), (27) that

$$\sum_{n \in \mathbb{Z}^+} \sum_{\nu=1}^M q_{(\tau(m)-1, \ell, \mu, n, \nu)} = 1 \quad (30)$$

for all $\ell, m \in \mathbb{Z}^{0+}$, $\mu, M \in \mathbb{Z}^{0+}$, and $\mu \leq M$.

Corollary 1.3: Under Assumption 1,

$$\begin{aligned} & \lim_{k \rightarrow \infty} q_{(k-1, \ell, \mu, n, \nu)} = q_{(\ell, \mu, n, \nu)} \\ &= \frac{\hat{e}_\mu^\top P_e^\top P_\pi P_s P_f^\ell P_\alpha(\nu) P_s P_f^n P_s \mathbf{1}}{\hat{e}_\mu^\top P_e^\top P_\pi P_s P_f^\ell P_s \mathbf{1}}, \end{aligned} \quad (31)$$

where $P_\pi = \bigoplus_{i=1}^N \pi_i$

Proof: A direct consequence of Assumption 1. ■

The following proposition guarantees that the state-space of the operational modes defined by (23) for $t \in \{1, 2\}$ is finite.

Proposition 2: $\forall \tau(m), \mu, \nu, M \in \mathbb{Z}^+$, with $\mu, \nu \leq M$, and an arbitrarily small threshold ϵ , there always exists a maximal number of consecutive control message dropouts L such that

$$L \triangleq \arg \min_{\hat{n} \in \mathbb{Z}^+} q_{(\tau(m)-1, 0, \mu, \hat{n}, \nu)} < \epsilon, \quad (32)$$

Proof: From (1)–(3), the matrix P_f is strictly sub-stochastic. From [16, Th. 8.1.22 and 5.6.12], it is convergent. For large enough values of $\hat{n}$, terms $q_{(\tau(m)-1, 0, \mu, \hat{n}, \nu)}$ become negligible. ■

Theorem 2: The system (7)–(16) subject to (1)–(5) is an HMJLS described by (17)–(32).

Proof: It follows from Proposition 1 and Theorem 1. ■

V. INTERNAL STABILITY

This section formally introduces the concept of, and the conditions for, the internal stability of the system (17)–(32). According to Theorem 2, these results apply to (7)–(16).

Definition 1: The system described by (17)–(32) is *internally stochastically stable* (ISS) whenever

$$\sum_{m=0}^{\infty} \mathbb{E}(\|x_{\tau(m)}\|^2) < \infty \quad (33)$$

for every initial condition x_0 with finite second moment and every initial Markov state $\varphi_{\tau(0)}$.

Definition 2: The system (17)–(32) is *stochastically stabilizable* (SS) if there exists K , as in (11), that makes the system ISS. We denote the set of such K by $\mathcal{K}$.

To write expressions concisely, we index the values of the ordered pairs (v_1, v_2) representing the operational modes of the closed-loop system, with $0 \leq v_1 \leq L$ and $1 \leq v_2 \leq M$, using an invertible mapping $f: \mathbb{Z}^{0+} \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$,

$$f(v_1, v_2) = Mv_1 + v_2. \quad (34)$$

For any $i, j \in \mathbb{Z}^+$ with $i, j \leq (L+1)M$, $(\ell, \hat{s}_\mu)$ is indexed by i if $f^{-1}(i) = (\ell, \mu)$ and $(n, \hat{s}_\nu)$ is indexed by j if $f^{-1}(j) = (n, \nu)$. Then, we can use, for example, the following representation of the transition probabilities in (31).

$$q_{(i,j)} = q_{(\ell,\mu,n,\nu)} \text{ such that } f^{-1}(i) = (\ell, \mu) \wedge f^{-1}(j) = (n, \nu). \\ \mathcal{M} \triangleq [q_{(i,j)}]_{i,j=1}^L, \quad L \triangleq (L+1)M. \quad (35)$$

Notice from Proposition 2, (1)–(6), and (27) that, $\forall m \in \mathbb{Z}^{0+}$,

$$\sum_{j=1}^L q_{(\tau_{(m)}-1, i, j)} = 1 - \epsilon. \quad (36)$$

Consider $\hat{u}_{\tau_{(m)}}$ as in (9). To derive the recursive difference equation for the second moment of the system's state, let $1_{\{\varphi_{\tau_{(m+1)}} = (\ell, \hat{s}_\mu)\}}$ be the indicator function that specifies the membership (or non-membership) of a given element in the set. For notational convenience, let $\mathcal{Q}_{\tau_{(m+1)}} \triangleq [Q_{(\tau_{(m+1)}, i)}]_{i=1}^L$,

$$Q_{(\tau_{(m+1)}, \ell, \mu)} \triangleq \mathbb{E} \left(x_{\tau_{(m+1)}} x_{\tau_{(m+1)}}^\top 1_{\{\varphi_{\tau_{(m+1)}} = (\ell, \hat{s}_\mu)\}} \right). \quad (37)$$

The second moment of (17)–(32) at time $\tau_{(m+1)}$ is then

$$\mathbb{E} \left(x_{\tau_{(m+1)}} x_{\tau_{(m+1)}}^\top \right) = \sum_{\ell \in \mathbb{Z}^{0+}} \sum_{\mu=1}^M \mathbb{E} \left(Q_{(\tau_{(m+1)}, \ell, \mu)} \right).$$

Proposition 3: $\forall \tau_{(m)}, \mu, \nu, M, L \in \mathbb{Z}^+$ and $\ell, m, n \in \mathbb{Z}^{0+}$, with $\ell, n \leq L$ as in (32), $\mu, \nu \leq M$, and $\tau_{(m)}$ as in (19),

$$Q_{(\tau_{(m+2)}, n, \nu)} = \Psi_{(n, \hat{s}_\nu)} \hat{\mathcal{E}}_{(n, \nu)}(\mathcal{Q}_{\tau_{(m+1)}}) \Psi_{(n, \hat{s}_\nu)}^\top, \quad (38)$$

$$\hat{\mathcal{E}}_{(n, \nu)}(\mathcal{Q}_{\tau_{(m+1)}}) \triangleq \sum_{\ell=0}^L \sum_{\mu=1}^M q_{(\tau_{(m)}-1, \ell, \mu, n, \nu)} Q_{(\tau_{(m+1)}, \ell, \mu)}.$$

Proof: It follows from the definitions of the expected value and the indicator function, the linearity of the expected value, and (27). For more technical details in various MJLS settings, see, for instance, [10], [17, Chapter 3], and [18]. ■

By Corollary 1.3, $\lim_{m \rightarrow \infty} q_{(\tau_{(m)}-1, \ell, \mu, n, \nu)} = q_{(\ell, \mu, n, \nu)}$. For any sequence of L square matrices $\mathcal{Y} \triangleq (Y_i)_{i=1}^L$, where each $Y_i \in \mathbb{R}^{n_x \times n_x}$, we define the following linear operator.

$$\mathcal{E}_j(\mathcal{Y}) \triangleq \sum_{i=1}^L q_{(i,j)} Y_i. \quad (39)$$

Let $\otimes$ denote the Kronecker product, vec the matrix vectorization, and vec^2 the matrix sequence vectorization.

$$\text{vec}^2(\mathcal{Y}) = \left[(\text{vec}(Y_1))^\top \cdots (\text{vec}(Y_L))^\top \right]^\top. \quad (40)$$

By Corollary 1.3 and Proposition 3,

$$\lim_{m \rightarrow \infty} Q_{(\tau_{(m+2)}, j)} = \Psi_j \mathcal{E}_j(\mathcal{Q}_{\tau_{(m+1)}}) \Psi_j^\top. \quad (41)$$

Furthermore, the vector form of (41) is

$$\text{vec}^2(\mathcal{Q}_{\tau_{(m+2)}}) = \Lambda \text{vec}^2(\mathcal{Q}_{\tau_{(m+1)}}), \quad (42)$$

where Λ is the stability verification matrix, and $I_{n_x^2}$ denotes the identity matrix of size n_x^2 .

$$\Lambda = \left(\bigoplus_{j=1}^L (\Psi_j \oplus \Psi_j) \right) (\mathcal{M}^\top \oplus I_{n_x^2}) = [(\Psi_j \oplus \Psi_j) q_{(j,i)}]_{i,j=1}^L \quad (43)$$

The following proposition, where ρ denotes the spectral radius of a square matrix, meaning the largest absolute value of its eigenvalues, explains the origin of the name for Λ .

Proposition 4: The system (17)–(32), with a given control strategy defined by (9)–(11), is ISS if and only if $\rho(\Lambda) < 1$.

Proof: After a transitory period, the second moment of the system evolves according to (42), where Λ , as defined by (43), is a fixed matrix. Therefore, the standard MJLS approach remains applicable, and following the steps of the proofs of [17, Th. 3.9] or [18, Th. 11 and 14] directly leads to the desired result. ■

The next proposition provides an equivalent ISS verification criterion based on Lyapunov-like coupled equations.

Proposition 5: The system (17)–(32), with a given control strategy defined by (9)–(11), is ISS if and only if there exists $\mathcal{Z} \triangleq (Z_j)_{j=1}^L$, where each $Z_j \in \mathbb{R}^{n_x \times n_x}$ and $Z_j \succ 0$, such that

$$Z_j \succ \Psi_j \mathcal{E}_j(\mathcal{Z}) \Psi_j^\top \quad (44)$$

are valid for all j .

Proof: Similar to the proof of Proposition 4, it relies on Corollary 1.3 and Proposition 3 to derive (41), which holds after a transitory period. The following steps use the standard MJLS approach from the proofs of [17, Th. 3.9] or [10, Th. 1] to obtain the desired result. ■

The next theorem establishes the foundation for LMI-based verification of the system's stabilizability.

Theorem 3: The system (17)–(32), with a given control strategy defined by (9)–(11), is ISS if there exist matrix sequences $\mathcal{X} = (X_j)_{j=1}^L$ and $\mathcal{V} = (V_\nu)_{\nu=1}^M$, where each $X_j, V_\nu \in \mathbb{R}^{n_x \times n_x}$ and $X_j = X_j^\top$, such that, $\forall \nu$ and $j = f(n, \nu)$,

$$\begin{bmatrix} X_{(n,\nu)} & \Psi_{(n,\nu)} V_\nu \\ \star & V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X}) \end{bmatrix} \succ 0. \quad (45)$$

Proof: To demonstrate sufficiency, assume (45) is true. From the Schur complement, $V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X}) \succ 0$ and $X_{(n,\nu)} - \Psi_{(n,\nu)} V_\nu (V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X}))^{-1} V_\nu^\top \Psi_{(n,\nu)}^\top \succ 0$. The first inequality implies that $V_\nu + V_\nu^\top \succ \mathcal{E}_{(n,\nu)}(\mathcal{X}) \succ 0$. As shown in [19], this means that $V_\nu \succ 0$ and

$$(\mathcal{E}_{(n,\nu)}(\mathcal{X}) - V_\nu)^\top \mathcal{E}_{(n,\nu)}(\mathcal{X})^{-1} (\mathcal{E}_{(n,\nu)}(\mathcal{X}) - V_\nu) \geq 0. \quad (46)$$

Since all X_j in $\mathcal{X}$ are symmetric, $\mathcal{E}_{(n,\nu)}(\mathcal{X})$ is consequently symmetric. Thus, (46) becomes

$$(I_{n_x} - V_\nu^\top \mathcal{E}_{(n,\nu)}(\mathcal{X})^{-1}) (\mathcal{E}_{(n,\nu)}(\mathcal{X}) - V_\nu) \geq 0,$$

implying that $V_\nu^\top \mathcal{E}_{(n,\nu)}(\mathcal{X})^{-1} V_\nu \geq V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X})$. For each n, ν , set $W_{(n,\nu)} = V_\nu^{-1} \mathcal{E}_{(n,\nu)}(\mathcal{X}) (V_\nu^{-1})^\top$, so that

$$W_{(n,\nu)}^{-1} = V_\nu^\top \mathcal{E}_{(n,\nu)}(\mathcal{X})^{-1} V_\nu \geq V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X}). \quad (47)$$

From (45), (47), and the Schur complement, we derive (44), where $\mathcal{Z} = \mathcal{X}$. Thus, by Proposition 5, the system is ISS. ■

Corollary 3.1: If there exist sequences of matrices $\mathcal{X} = (X_j)_{j=1}^L$, $\mathcal{V} = (V_\nu)_{\nu=1}^M$, and $\mathcal{U} = (U_\nu)_{\nu=1}^M$ of appropriate sizes, where each $X_j = X_j^\top$, such that, for all ν and $j = f(n, \nu)$,

$$\begin{bmatrix} X_{(n,\nu)} & \mathcal{A}_n V_\nu + \mathcal{B}_n U_\nu \\ \star & V_\nu + V_\nu^\top - \mathcal{E}_{(n,\nu)}(\mathcal{X}) \end{bmatrix} \succ 0, \quad (48)$$

then the system (17)–(32) is stochastically stabilizable with $K_\nu = U_\nu V_\nu^{-1}$.

TABLE I
SIZE OF A

M	1	2	4
This paper	448	896	1792
[6]	448	1792	7168

Proof: It is a direct result of Theorem 3 and (26): $\Psi_{(n,\nu)}V_\nu = (A_n + B_nK_\nu)V_\nu$, $U_\nu = K_\nu V_\nu$, and $V_\nu \succ 0$. ■
The following section verifies these results.

VI. NUMERICAL EXAMPLE

Consider a linearized model of a rotary inverted pendulum as described in [4]:

$$A = \begin{bmatrix} 1 & 0.224 & 0.055 & 0.004 \\ 0 & 1.369 & -0.028 & 0.090 \\ 0 & 4.994 & 0.391 & 0.167 \\ 0 & 8.618 & -0.634 & 1.270 \end{bmatrix}, \quad B = \begin{bmatrix} 0.227 \\ 0.218 \\ 4.944 \\ 4.820 \end{bmatrix}.$$

The controller transmits messages to actuators via a wireless link modeled as the following FSMC.

$$P_c = \begin{bmatrix} 0.257 & 0.027 & 0.032 & 0.684 \\ 0.182 & 0.023 & 0.028 & 0.767 \\ 0.172 & 0.022 & 0.027 & 0.779 \\ 0.058 & 0.010 & 0.012 & 0.920 \end{bmatrix}, \quad \hat{\delta} = \begin{bmatrix} 0.026 \\ 0.375 \\ 0.634 \\ 0.995 \end{bmatrix}^\top,$$

As in [6], we set $\Phi=0.1$. We verified the internal stochastic stability of all the optimal infinite-horizon linear-quadratic regulators derived there, but in much less time, thanks to smaller stability verification matrices and fewer computations needed to find the transition probabilities. Table I compares the A size in [6], $n_x^2 M^2(L+1)$, and this paper's $n_x^2 M(L+1)$ for each selected scenario, where $P_e = 1$ indicates no CSI case ($M = 1$), $P_e = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^\top$ represents the case of partial observation of CSI using clusters ($M = 2$), and $P_e = I_4$ denotes the perfect CSI scenario ($M = 4$). As a reference, computing $\rho(A)$ for $M = 4$ on the same computer takes less than a second with the method proposed in this paper, whereas it takes over 98 seconds with the method in [6]. Both methods yield the same results (for example, $\rho(A) = 0.999999552386$ for $n_f = 0$ and $\rho(A) = 0.847248234365$ for $n_f = 1$). This data emphasizes the practical significance of the presented approach for verifying internal stochastic stability. Finally, we tested the feasibility of LMIs (48) in the MATLAB Robust Control Toolbox [20] using *feasp*. As a reference, establishing stochastic stabilizability via (48) took longer than 290 seconds for $n_f = 1$, yielding gains with $\rho(A) = 0.576992328678$, and over 485 seconds for $n_f = 0$, yielding gains with $\rho(A) = 0.9999999972$. The feasibility of (48) for such high $\rho(A)$ values shows that this sufficient condition is not overly conservative, making it especially useful for integration into $\mathcal{H}_\infty$ control synthesis problems.

VII. CONCLUSIONS

This paper proposed an advanced hidden Markov jump linear system model for discrete-time linear stochastic systems that includes hybrid message dropout compensation and imperfect wireless channel state information. The model proved effective in analyzing stabilizability and internal stability. Additionally, LMI formulations were developed to verify

stochastic stabilizability and to design stabilizing feedback controls, and the results were validated through a numerical example. Future research may focus on applying this model and framework to address $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control synthesis problems.

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