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Robust stability of time-inhomogeneous Markov jump linear systems ⋆

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Abstract: In this work we derive necessary and sufficient conditions for robust mean square stability of discrete-time time-inhomogeneous Markov jump linear systems (MJLSs) affected by polytopic uncertainties on transition probabilities and bounded disturbances.

1. INTRODUCTION

Wireless control networks (WCN) are distributed control systems where the communication between sensors, actuators, and computational units is supported by a wireless communication network. The use of WCN in industrial automation results in flexible architectures and generally reduces installation, debugging, diagnostic and maintenance costs with respect to wired networks (see e.g. Akyildiz and Kasimoglu (2004) and references therein). However modeling, analysis and design of (wireless) networked control systems (NCSs) are challenging open research problems since they require to take into account the joint dynamics of physical systems, communication protocols and network infrastructures. Recently, a huge effort has been made in scientific research on NCSs, see e.g. Hespanha et al. (2007), Schenato et al. (2007), Gupta et al. (2009), Donkers et al. (2011), Pajic et al. (2011), Alur et al. (2011), D'Innocenzo et al. (2013) and references therein for a general overview. In this domain it has been shown (e.g. in Schenato et al. (2007), Gonçalves et al. (2010), Smarra et al. (2015), Di Girolamo et al. (2015)) that discrete-time Markov-jump linear systems (MJLS, Costa et al. (2005)) represent a promising mathematical model to jointly take into account the dynamics of a physical plant and non-idealities of wireless communication such as packet losses. A MJLS is, basically, a switching linear system where the switching signal is a Markov chain. The transition probability matrix (TPM) of the Markov chain can be used to model the stochastic process that rules packet losses due to wireless communication. However, in most real cases, such probabilities cannot be computed exactly and are time-varying. We can take into account this aspect by assuming that the Markov chain of a MJLS is polytopic time-inhomogeneous (PTI), i.e. a Markov chain having its TPM varying over time, with variations that are arbitrary within a polytopic set of stochastic matrices. Given such mathematical model, the first problem to be addressed is the (mean square) stability problem. Some recent work addressed the above problem: in Aberkane (2011) a sufficient condition for stochastic stability in terms of linear matrix inequality feasibility problem is provided, while in Chitraganti et al. (2013) a sufficient condition for mean square stability (MSS) of system with interval TPM, which in turn can be represented as a convex polytope (see Hartfiel (1998) for additional details), is presented in relation to spectral radius; in general, only sufficient stability conditions have been derived for MJLS with PTI Markov chains having TPM arbitrarily varying within a polytopic set of stochastic matrices. In Zacchia Lun et al. (2016) we provide necessary and sufficient conditions for MSS of discrete-time MJLS with time-inhomogeneous Markov chains.

In this paper we extend such results deriving necessary and sufficient conditions for robust mean square stability of a discrete-time time-inhomogeneous MJLSs affected not only by polytopic uncertainties on transition probabilities but also by bounded disturbances. Such conditions require to decide whether the joint spectral radius (JSR) of a finite family of matrices is smaller than 1.

2. NOTATION AND CONCEPTUAL PRELIMINARIES

The *notation* used throughout is standard. The sets of all positive and nonnegative integers are represented by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The set of first k nonnegative integers is denoted by $\mathbb{N}_k$, i.e. $\mathbb{N}_k \triangleq \{i \in \mathbb{N}_0; i \leq k\}$, $\forall k \in \mathbb{N}_0$. If X is a normed linear space (an inner product space), then the symbols $\|\cdot\|$ and $\langle \cdot ; \cdot \rangle$ stand for norm and inner product in X, respectively. If X and Y are normed linear spaces, then $\mathcal{B}[X,Y]$ denotes the normed linear space of all bounded linear transformations of X into Y. For simplicity we set $\mathcal{B}[X] \triangleq \mathcal{B}[X,X]$. Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and $\mathbb{F}^n$ the n-dimensional (either real or complex) Euclidean space. A transformation in $\mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$ will be identified with its $m \times n$ matrix representation relative to the standard orthonormal bases for $\mathbb{F}^n$ and $\mathbb{F}^m$. The conjugate of a complex matrix is denoted by overbar $\bar{\cdot}$, while the superscript * indicates the conjugate transpose of a matrix, and ^{T} indicates the transpose. Clearly for a set of real matrices, * and ^{T} have the same meaning. We indicate with $\mathbb{C}_*^{n\times n}$ the set of Hermitian matrices, and with $\mathbb{F}_+^{n\times n}$ the set of positive semi-definite matrices. The $n \times n$ identity matrix is denoted by $\mathbb{I}_n$. For arbitrary row vectors $x \in \mathbb{F}^m$, $y, z \in \mathbb{F}^n$, the transformation $(xy^*) \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$, such that $(xy^*)z = x(y^*z) = x\langle z; y\rangle = x\sum_{i=1}^{n} z_i \bar{y}_i$, $\forall z$,

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is identified with the usual outer product $m \times n$ matrix $[x_i \bar{y}_j]$, $i = 1, \dots, m$, $j = 1, \dots, n$. Unless otherwise stated, $\|\cdot\|$ will indicate any norm in $\mathbb{F}^n$, and, $\forall M \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$, $\|M\|$ will denote the induced uniform norm in $\mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$. We will use of the trace operator $\operatorname{tr}(\cdot): \mathcal{B}[\mathbb{F}^n] \to \mathbb{F}$, defined on elements m_{ij} of $M \in \mathbb{F}^{n \times n}$ as $\operatorname{tr}[M] = \sum_{i=1}^n m_{ii}$. The trace operator has a commutative property, that is $\operatorname{tr}(KL) = \operatorname{tr}(LK)$. The linear space made up of all N sequences $\boldsymbol{M} = (M_1, \dots, M_N)$, with $M_i \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$, $i \in \mathcal{N}$, is indicated by $\mathbb{H}^{n,m}$, where $\mathcal{N} \triangleq \{1, \dots, N\}$ is a finite set. For simplicity, we set $\mathbb{H}^n \triangleq \mathbb{H}^{n,n}$. For any $\boldsymbol{M} \in \mathbb{H}^{n,m}$, we define the following equivalent norms in the finite dimensional space $\mathbb{H}^{n,m}$: $\|\boldsymbol{M}\|_{\max} \triangleq \max\{\|M_i\|; i \in \mathcal{N}\}$, $\|\boldsymbol{M}\|_1 \triangleq \sum_{i=1}^N \|M_i\|$, and $\|\boldsymbol{M}\|_2 \triangleq \sqrt{\sum_{i=1}^N \operatorname{tr}(M_i^* M_i)}$. We shall omit the subscripts 1, 2, max whenever the definition of a specific norm does not affect the result being considered. It is easy to verify that $\mathbb{H}^{n,m}$ equipped with any of the above norms is a Banach space and, in fact, $(\|\cdot\|_2, \mathbb{H}^{n,m})$ is a Hilbert space (Costa et al., 2005, p. 16), with the inner product given, for $\boldsymbol{M}, \boldsymbol{V} \in \mathbb{H}^{n,m}$, by $\langle \boldsymbol{M}; \boldsymbol{V} \rangle = \sum_{i=1}^N \operatorname{tr}(M_i^* V_i)$. For $\boldsymbol{M} \in \mathbb{H}^{n,m}$ we write $\boldsymbol{M}^* = (M_1^*, \dots, M_N^*) \in \mathbb{H}^{m,n}$, and say that $\boldsymbol{M} \in \mathbb{H}^n$ is Hermitian if $\boldsymbol{M} = \boldsymbol{M}^*$. We denote $\mathbb{H}^{n*} \triangleq \{\boldsymbol{M} \in \mathbb{H}^n; M_i = M_i^*, i \in \mathcal{N}\}$, $\mathbb{H}^{n+} \triangleq \{\boldsymbol{M} \in \mathbb{H}^{n*}; M_i \geq 0, i \in \mathcal{N}\}$. We write, $\forall \boldsymbol{M}, \boldsymbol{V} \in \mathbb{H}^n$, that $\boldsymbol{M} \geq \boldsymbol{V}$, when $\boldsymbol{M} - \boldsymbol{V} = (M_1 - V_1, \dots, M_N - V_N) \in \mathbb{H}^{n+}$, and that $\boldsymbol{M} > \boldsymbol{V}$, if $M_i - V_i > 0$, $\forall i \in \mathcal{N}$. We use the vectorization transformation (Neudecker (1969)), defined $\forall M \in \mathcal{B}[\mathbb{F}^n, \mathbb{F}^m]$ as $\varphi(M) \triangleq \operatorname{vec}(M)$, where, indicating with $(M)_{\bullet j}$ the j-th column of $M \in \mathbb{F}^{m \times n}$, we have that

$$\operatorname{vec}(M) \triangleq \begin{bmatrix} M_{\bullet 1} \\ \vdots \\ M_{\bullet n} \end{bmatrix} \in \mathbb{C}^{mn}, \ \hat{\varphi}[\boldsymbol{M}] \triangleq \begin{bmatrix} \varphi(M_1) \\ \vdots \\ \varphi(M_N) \end{bmatrix} \in \mathbb{C}^{Nmn} \quad (1)$$

Remark 1. The spaces $\mathbb{H}^{n,m}$ and $\mathbb{C}^{Nmn}$ are uniformly homeomorphic (Naylor and Sell, 2000, p. 117) through the linear mapping $\hat{\varphi}$ (Costa et al., 2005, p. 17).

Finally, $\mathbf{E}[\cdot]$ stands for the mathematical expectation of the underlying scalar valued random variables.

3. JOINT SPECTRAL RADIUS

The results of this paper use the notion of joint spectral radius (JSR, Rota and Strang (1960)), which in the last decades has been subject of intense research due to its role in the study of wavelets, switching systems, approximation algorithms, and many other topics (Jungers (2009)).

Let $\mathcal{M}$ be a family of square matrices, i.e. $\mathcal{M} = \{M_l\}_{l \in \mathcal{L}}$, where $M_l \in \mathbb{F}^{n \times n}$, $\mathcal{L} \triangleq \{1, ..., L\}$. For each $k \in \mathbb{N}$, consider the set $\Pi_k(\mathcal{M})$ of all possible products of length k whose factors are elements of $\mathcal{M}$, that is

$$\Pi_k(\mathcal{M}) = \left\{ \left(\prod_{i=1}^k M_{l_i}^* \right)^* \mid l_1, \dots, l_k \in \mathcal{L} \right\}$$

Definition 2. (Joint spectral radius, JSR). For any matrix norm $\|\cdot\|$ on $\mathbb{F}^{n \times n}$, consider the supremum among the normalized norms of all products in $\Pi_k(\mathcal{M})$, i.e.

$$\hat{\rho}_k(\mathcal{M}) \triangleq \sup_{\Pi \in \Pi_k(\mathcal{M})} \|\Pi\|^{1/k}, \ k \in \mathbb{N}$$

The joint spectral radius of $\mathcal{M}$ is defined as

$$\hat{\rho}(\mathcal{M}) = \lim_{k \to \infty} \hat{\rho}_k(\mathcal{M})$$

The JSR of a bounded set of matrices has some interesting properties reported below.

Proposition 3. (Convex hull). The convex hull of a set has the same joint spectral radius as the original set, i.e.

$$\hat{\rho}(\operatorname{conv} \mathcal{M}) = \hat{\rho}(\mathcal{M})$$

Proof. See Barabanov (1988, 2005), Cicone (2015). □

Proposition 4. (Convergence of matrix products). For any bounded set of matrices $\mathcal{M}$ and for any $k \in \mathbb{N}$, all matrix products $\Pi \in \Pi_k(\mathcal{M})$ converge to zero matrix as $k \to \infty$, if and only if $\hat{\rho}(\mathcal{M}) < 1$.

Proof. See (Berger and Wang, 1992, Theorem I (b)). □

Remark 5. The concept of JSR was introduced for a bounded subset of any normed algebra. In fact, Rota and Strang (1960) presented JSR also in a special case of a subalgebra of the algebra of bounded operators on a Banach space, providing an alternative construction of the norms. Combining this consideration with the one from Remark 1, we can state that any bounded subset $\mathcal{Z}$ of operators in $\mathcal{B}[\mathbb{H}^{n,m}]$ can be represented in $\mathcal{B}[\mathbb{C}^{Nmn}]$ through the linear mapping $\hat{\varphi}[\mathcal{Z}]$, with

$$\hat{\rho}(\mathcal{Z}) = \hat{\rho}(\hat{\varphi}[\mathcal{Z}])$$

4. PROBABILITY SPACE WITH POLYTOPIC TRANSITION MATRIX

In order to define the mathematical model we consider in this paper, i.e. the discrete-time time-inhomogeneous MJLS with polytopic uncertainties in the TPM, we need some preliminary technical definitions: let us consider a probability space $(\Omega, \mathcal{F}, \Pr)$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of events and Pr is the probability measure. Let $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ be a Markov chain defined on the probability space, which takes values in a finite set $\mathcal{N}$. For $k \in \mathbb{N}_0$ we define the transition probability as

$$p_{ij}(k) = \Pr\{\theta(k+1) = j \mid \theta(k) = i\} \geq 0, \ \sum_{j=1}^N p_{ij}(k) = 1.$$

The associated TPM $P(k)$ is a stochastic $N \times N$ matrix with entries $p_{ij}(k)$. In this work we assume that $P(k)$ is unknown and time-varying within a bounded set.

Assumption 6. TPM $P(k)$ is **polytopic**, i.e. $\forall k \in \mathbb{N}_0$

$$P(k) = \sum_{l=1}^L \lambda_l(k) P_l, \ \lambda_l(k) \geq 0, \ \sum_{l=1}^L \lambda_l(k) = 1, \quad (2)$$

where $\{P_l\}_{l \in \mathcal{L}} \triangleq \mathcal{P}_L$ is a given set of TPMs, which are the vertices of a convex polytope, $\lambda_l(k)$ are unmeasurable.

Remark 7. The Assumption 6 is not restrictive, since the polytopic uncertainty model is widely used for robust control of time-homogeneous MJLS (see e.g. Gonçalves et al. (2011)) and is considered to be more general than the partly known element model of TPM uncertainties; furthermore, also the interval TPM can be represented as a convex polytope (Hartfiel (1998)).

We set $C^m = L_2(\Omega, \mathcal{F}, \Pr, \mathbb{C}^m)$ the Hilbert space of all $\mathbb{C}^m$-valued $\mathcal{F}$-measurable random variables with inner product $\langle x; y \rangle = \mathbf{E}[x^* y]$, and norm $\|\cdot\|_2$. We set $\ell_2(C^m) = \oplus_{k \in \mathbb{N}_0} C^m$, the direct sum of countably infinite copies of C^m, which is a Hilbert space made up of $\mathfrak{z} = \{\mathfrak{z}(k); k \in \mathbb{N}_0\}$, $\mathfrak{z}(k) \in C^m$, s.t. $\|\mathfrak{z}\|_2^2 \triangleq \sum_{k \in \mathbb{N}_0} \mathbf{E}[\|\mathfrak{z}(k)\|^2] < \infty$. For $\mathfrak{z}, \mathfrak{u} \in \ell_2(C^m)$, the inner product is $\langle \mathfrak{z}; \mathfrak{u} \rangle \triangleq \sum_{k \in \mathbb{N}_0} \mathbf{E}[\mathfrak{z}^*(k) \mathfrak{u}(k)] \leq \|\mathfrak{z}\|_2 \|\mathfrak{u}\|_2$. We define $\mathcal{C}^m \subset \ell_2(C^m)$ as follows: $\mathfrak{z} = \{\mathfrak{z}(k); k \in \mathbb{N}_0\} \in \mathcal{C}^m$ if

$\mathfrak{z}\in\ell_2(C^m)$ and $\mathfrak{z}(k)\in L_2(\Omega,\mathcal{F}_k,\Pr,\mathbb{C}^m)\ \forall k\in\mathbf{N}_0$, where $\mathcal{F}_k$ is a σ-algebra generated by $i\in\mathbb{N}_k$ events. Clearly, $\forall k\in\mathbb{N}_0$,

$$\mathcal{F}_k\subset\mathcal{F}_{k+1}\subset\mathcal{F}.$$

We have that $\mathcal{C}^m$ is a closed linear subspace of $\ell_2(C^m)$ and therefore a Hilbert space (Costa et al., 2005, p.21).
We define $\mathcal{C}_k^m$ as formed by elements $\mathfrak{z}_k=\{\mathfrak{z}(k);k\in\mathbb{N}_k\}$, s.t. $\mathfrak{z}(l)\in L_2(\Omega,\mathcal{F}_l,\Pr,\mathbb{C}^m)$, $\forall l\in\mathbb{N}_k$. Finally, Θ_0 is the set of all $\mathcal{F}_0$-measurable variables taking values in $\mathcal{N}$.

5. STABILITY OF AUTONOMOUS DISCRETE-TIME PTI MJLS WITH BOUNDED NOISE

Let us consider an autonomous discrete-time polytopic time-inhomogeneous MJLS $(\mathcal{S})$ described by

$$\begin{cases}x(k+1)=A_{\theta(k)}x(k)+G_{\theta(k)}w(k),\\ x(0)=x_0,\ \theta(0)=\theta_0,\end{cases}\tag{3}$$

where $k\in\mathbb{N}_0$ is a time step, $x(k)\in\mathbb{F}^n$ is the state vector, and $w(k)\in\mathbb{F}^r$ is an additive disturbance representing the process noise. The set $\mathcal{N}$ comprises the operational modes of the system $(\mathcal{S})$ and, for each possible value of $\theta(k)=i$, $i\in\mathcal{N}$, we denote each matrix associated with the i-th mode by e.g. $A_i=A_{\theta(k)=i}$. Thus, $\boldsymbol{A}=(A_1,\dots,A_N)\in\mathbb{H}^n$ and $\boldsymbol{G}=(G_1\dots,G_N)\in\mathbb{H}^{r,n}$ are vectors of state and process noise matrices, respectively, each of which is associated with an operational mode of the system. Finally, $x(0)\in\mathcal{C}_0^n$ and $\theta(0)\in\Theta_0$ are initial conditions.
It is easy to see that the system state evolves as

$$x(k)=\left(\prod_{i=0}^{k-1}A^*_{\theta(i)}\right)^*x(0)+\sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{k-1}A^*_{\theta(j)}\right)^*G_{\theta(i)}w(i).\tag{4}$$

For a set $\Theta\in\mathcal{F}$, we define the indicator function $\mathbf{1}_\Theta$ in the usual way, that is, $\forall\omega\in\Omega$,

$$\mathbf{1}_\Theta(\omega)=\begin{cases}1 & \text{if }\omega\in\Theta,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that, $\forall i\in\mathbb{N}$,

$$\mathbf{1}_{\{\theta(k)=i\}}(\omega)=1\text{ if }\theta(k)(\omega)=i,\text{ and }0\text{ otherwise.}$$

$$\mathbf{E}[x(k)]=\sum_{i=1}^N\mathbf{E}[x(k)\mathbf{1}_{\{\theta(k)=i\}}],$$

$$\mathbf{E}[x(k)x^*(k)]=\sum_{i=1}^N\mathbf{E}[x(k)x^*(k)\mathbf{1}_{\{\theta(k)=i\}}].$$

Following the standard workflow for MJLSs (Costa et al., 2005, p. 31), we use the subsequent notation:

$$q_i(k)\triangleq\mathbf{E}[x(k)\mathbf{1}_{\{\theta(k)=i\}}]\in\mathbb{F}^n,\tag{5}$$

$$\boldsymbol{q}(k)\triangleq[q_1(k),\cdots,q_N(k)]^T\in\mathbb{F}^{Nn},$$

$$r_i(k)\triangleq\mathbf{E}[w(k)\mathbf{1}_{\{\theta(k)=i\}}]\in\mathbb{F}^r,\tag{6}$$

$$\boldsymbol{r}(k)\triangleq[r_1(k),\cdots,r_N(k)]^T\in\mathbb{F}^{Nr},$$

$$Q_i(k)\triangleq\mathbf{E}[x(k)x^*(k)\mathbf{1}_{\{\theta(k)=i\}}]\in\mathcal{B}[\mathbb{F}^n]^+,\tag{7}$$

$$\boldsymbol{Q}(k)\triangleq(Q_1(k),\dots,Q_N(k))\in\mathbb{H}^{n+},\tag{8}$$

$$W_i(k)\triangleq\mathbf{E}[w(k)w^*(k)\mathbf{1}_{\{\theta(k)=i\}}]\in\mathcal{B}[\mathbb{F}^r]^+,\tag{9}$$

$$\boldsymbol{W}(k)\triangleq(W_1(k),\dots,W_N(k))\in\mathbb{H}^{r+},$$

$$\boldsymbol{GW}(k)\boldsymbol{G}^*\triangleq(G_1W_1(k)G_1^*,\dots,G_NW_N(k)G_N^*)\in\mathbb{H}^{n+},$$

$$\mathcal{X}_i(k)\triangleq\mathbf{E}[x(k)w^*(k)\mathbf{1}_{\{\theta(k)=i\}}]\in\mathcal{B}[\mathbb{F}^r,\mathbb{F}^n],\tag{10}$$

$$\mathcal{X}(k)\triangleq(\mathcal{X}_1(k),\dots,\mathcal{X}_N(k))\in\mathbb{H}^{r,n},$$

$$\boldsymbol{A}\mathcal{X}(k)\boldsymbol{G}^*\triangleq(A_1\mathcal{X}_1(k)G_1^*,\dots,A_N\mathcal{X}_N(k)G_N^*)\in\mathbb{H}^{n+}\tag{11}$$

This permits us to define the expected value of $x(k)$ as

$$\mu(k)\triangleq\mathbf{E}[x(k)]=\sum_{i=1}^N q_i(k)\in\mathbb{F}^n,$$

and the second moment of $x(k)$ as

$$\mathcal{Q}(k)\triangleq\mathbf{E}[x(k)x^*(k)]=\sum_{i=1}^N Q_i(k)\in\mathcal{B}[\mathbb{F}^n]^+.\tag{12}$$

We can easily see that the recursive equations for $q_i(k)$ and $Q_i(k)$ in the polytopic time-inhomogeneous case with bounded disturbance have the same structure as the time-homogeneous case with known probability matrix (Costa et al., 2005, p. 32), and the extension to this more general case is done in the following manner.

Proposition 8. Consider the system $(\mathcal{S})$. $\forall k\in\mathbb{N}_0$, $j\in\mathcal{N}$

$$q_j(k+1)=\sum_{i=1}^N p_{ij}(k)A_iq_i(k)+\sum_{i=1}^N p_{ij}(k)G_ir_i(k),$$

$$Q_j(k+1)=\sum_{i=1}^N p_{ij}(k)A_iQ_i(k)A_i^*+\sum_{i=1}^N p_{ij}(k)G_iW_i(k)G_i^*+2\,\mathrm{Re}\left[\sum_{i=1}^N p_{ij}(k)A_i\mathcal{X}_i(k)G_i^*\right],$$

with $\mathrm{Re}[\cdot]$ indicating the real part of a complex matrix.

Proof. Regarding the first statement, from (5), (3) and (6), by linearity of the expected value, we have that

$$q_j(k+1)=\sum_{i=1}^N\mathbf{E}[(A_ix(k)+G_iw(k))\,\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}]$$
$$=\sum_{i=1}^N p_{ij}(k)A_iq_i(k)+\sum_{i=1}^N p_{ij}(k)G_ir_i(k).$$

The second statement can be proven similarly, from (7), (3), (9), (10) and (11). □

To rewrite the recursive equations for $Q_i(k)$ in matrix form, let us first focus for simplicity on $(\mathcal{S})$ in the **noiseless case** and consider a useful result regards the inequality between the $\|\boldsymbol{q}(k)\|$ and $\|\boldsymbol{Q}(k)\|_1$ (Costa et al., 2005, p. 35, within the proof of Proposition 3.6).

Proposition 9. Consider the noiseless version of system $(\mathcal{S})$, i.e. with $w(k)=0,\forall k\in\mathbb{N}_0$. Then

$$\|\boldsymbol{q}(k)\|^2\le n\|\boldsymbol{Q}(k)\|_1,\ \forall k\in\mathbb{N}_0.\tag{13}$$

We denote by $\otimes$ a Kronecker product defined in the usual way (Brewer (1978)). For any X,Y,Z,M given matrices of appropriate size, the following properties are satisfied:

$$(X+Y)\otimes(Z+M)=X\otimes Z+Y\otimes Z+X\otimes M+Y\otimes M\tag{14a}$$

$$\varphi(XYZ)=(Z^T\otimes X)\varphi(Y)\tag{14b}$$

As for time-homogeneous noiseless case (Costa et al., 2005, pp. 33-35), also here, via application of (7), Proposition 8 (where the second and third summations in the expression of $Q_j(k+1)$ are equal to zero, see Zacchia Lun et al. (2016) for additional details), (1) and (14) to (8), we have that

$$\hat{\varphi}(\boldsymbol{Q}(k+1))=\Lambda(k)\hat{\varphi}(\boldsymbol{Q}(k)),\tag{15}$$

$$\Lambda(k)\triangleq(P^T(k)\otimes\mathbb{I}_{n^2})diag[\bar{A}_i\otimes A_i],\quad\Lambda(k)\in\mathbb{F}^{Nn^2\times Nn^2},$$

$$diag[\bar{A}_i\otimes A_i]\triangleq\begin{bmatrix}\bar{A}_1\otimes A_1 & 0 & \cdots & 0\\ 0 & \bar{A}_2\otimes A_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{A}_N\otimes A_N\end{bmatrix}.$$

Consequently, from (15) we have that

$$\hat{\varphi}(\boldsymbol{Q}(k))=\left(\prod_{i=0}^{k-1}\Lambda^*(i)\right)^*\hat{\varphi}(\boldsymbol{Q}(0))=\Lambda\,\hat{\varphi}(\boldsymbol{Q}(0)).\tag{16}$$

Proposition 10. The matrix $\Lambda(k)$ associated to the second moment of $x(k)$ is polytopic, i.e. $\forall k\in\mathbb{N}_0$, $\forall l\in\mathcal{L}$

$$\Lambda(k) = \sum_{l=1}^{L} \lambda_l(k)\Lambda_l,\ \lambda_l(k) \geq 0,\ \sum_{l=1}^{L} \lambda_l(k) = 1,$$
$$\Lambda_l \triangleq (P_l^T \otimes \mathbb{I}_{n^2})diag[\bar{A}_i \otimes A_i],\ P_l \in \mathcal{P}_L \quad (17)$$

Proof. Direct application of (2) and (14a) to (17). □

Let $\mathcal{A}_L$ be the set $\{\Lambda_l\}_{l\in\mathcal{L}}$ of all vertices of the polytope associated to the second moment of $x(k)$.

Remark 11. Recalling the definition of the convex hull (Grünbaum, 2003, p. 14), we can write that $\forall k \in \mathbb{N}_0$, $\Lambda(k)\in \operatorname{conv}\mathcal{A}_L$.

We can now consider $(\mathcal{S})$ **with noise** and, still applying (7), Proposition 8, (1) and (14) to (8), rewrite the recursive equations for $Q_i(k)$ in the following matrix form:

$$\hat{\varphi}(\boldsymbol{Q}(k+1)) = \Lambda(k)\hat{\varphi}(\boldsymbol{Q}(k)) + \Gamma(k)\hat{\varphi}(\boldsymbol{W}(k)) + 2\mathrm{Re}[\Xi(k)\hat{\varphi}(\mathcal{X}(k))], \quad (18)$$
$$\Gamma(k) \triangleq (P^T(k) \otimes \mathbb{I}_{n^2})diag[\bar{G}_i \otimes G_i],$$
$$\Xi(k) \triangleq (P^T(k) \otimes \mathbb{I}_{n^2})diag[\bar{G}_i \otimes A_i].$$

It follows immediately from (18) and (16) that

$$\hat{\varphi}(\boldsymbol{Q}(k)) = \Lambda\,\hat{\varphi}(\boldsymbol{Q}(0)) + \sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{k-1}\Lambda^*(i)\right)^*\Gamma(i)\hat{\varphi}(\boldsymbol{W}(i)) + 2\mathrm{Re}\left[\sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{k-1}\Lambda^*(i)\right)^*\Xi(i)\hat{\varphi}(\mathcal{X}(i))\right].$$

Now we are ready to state our result on the boundedness of the process state.

Theorem 12. Given a discrete-time MJLS $(\mathcal{S})$ as in (3) with unknown and time-varying TPM $P(k) \in \operatorname{conv}\mathcal{P}_L$, then $\hat{\rho}(\mathcal{A}_L) < 1$ if and only if $x = \{x(k); k \in \mathbb{N}_0\} \in \mathcal{C}^n$ for every $w=\{w(k); k \in \mathbb{N}_0\} \in \mathcal{C}^r$, $x_0 \in \mathcal{C}_0^n$ and $\theta_0 \in \Theta_0$.

Proof. To prove necessity (that is, $\hat{\rho}(\mathcal{A}_L) < 1 \Rightarrow x \in \mathcal{C}^n$ $\forall w \in \mathcal{C}^r$, $x_0 \in \mathcal{C}_0^n$, $\theta_0 \in \Theta_0$), all we have to show is that $\|x\|_2 < \infty$ since clearly $x_k = (x(0), \dots, x(k)) \in \mathcal{C}_k^n$ $\forall k\in\mathbb{N}_0$. Using the triangular inequality in $\mathcal{C}^n$ on (4), we obtain

$$\|x(k)\|_2 \leq \left\|\left(\prod_{i=0}^{k-1}A^*_{\theta(i)}\right)^*x(0)\right\|_2 + \sum_{i=0}^{k-1}\left\|\left(\prod_{j=i+1}^{k-1}A^*_{\theta(j)}\right)^*G_{\theta(i)}w(i)\right\|_2 \quad (19)$$

Let us consider the *first term* of the right hand side of (19), which is clearly related to the noiseless version of system $(\mathcal{S})$ as in (3), i.e. when $w(k)=0$ $\forall k\in\mathbb{N}_0$. Since $\hat{\rho}(\mathcal{A}_L)<1$,

$$\lim_{k\to\infty}\left\|\left(\prod_{i=0}^{k}\Lambda^*(i)\right)^*\right\|^{1/k} = \hat{\rho}(\mathcal{A}_L),$$

by the radical test for infinite series, we can state that

$$\|\Lambda\| < \zeta^k,\quad \forall k \geq k',\quad \forall\zeta \in (\hat{\rho}(\mathcal{A}_L), 1),\quad \text{for some } k'\in\mathbb{N}_0.$$

With $\beta' = \zeta^{-k'} \sup_{\Pi\in\Pi_j[\mathcal{A}_L],\,0\leq j\leq k'} \|\Pi\|$, $\beta' \geq 1$, we have

$$\|\Lambda\| \leq \beta'\zeta^k,\ \forall k \in \mathbb{N}_0. \quad (20)$$

From the definition of the trace operator, we have that

$$\mathbf{E}[\|x(k)\|^2] = \mathbf{E}[\operatorname{tr}(x(k)x^*(k))] = \sum_{\iota=1}^{N}\operatorname{tr}(\mathbf{E}[x(k)x^*(k)\mathbf{1}_{\{\theta(k)=\iota\}}]) = \sum_{\iota=1}^{N}\operatorname{tr}(Q_\iota(k)) \leq n\sum_{\iota=1}^{N}\|Q_\iota(k)\|.$$

Observing that if X is a finite-dimensional linear space, then any two norms on X are equivalent (Kubrusly, 2001, Theorem 4.27), we will make use of Frobenius norm, defined for a matrix $M \in \mathbb{F}^{m,n}$ as

$$\|M_\iota\|_F \triangleq \sqrt{\operatorname{tr}(M_\iota^* M_\iota)} = \|\operatorname{vec}(M_\iota)\|_F = \|\varphi(M_\iota)\|_F.$$

Let us indicate by $\Lambda_{[(\iota-1)n^2+1,\iota n^2]\bullet}$ a matrix obtained by taking n consecutive rows (starting from $((\iota-1)n^2+1)$-th row, $\iota=1,\dots,N$) of Λ. Then,

$$\|Q_\iota(k)\|_F = \|\varphi(Q_\iota(k))\|_F = \|\Lambda_{[(\iota-1)n^2+1,\iota n^2]\bullet}\hat{\varphi}(\boldsymbol{Q}(0))\|_F \leq \|\Lambda_{[(\iota-1)n^2+1,\iota n^2]\bullet}\|_F\|\hat{\varphi}(\boldsymbol{Q}(0))\|_F \leq \|\Lambda\|_F\|\hat{\varphi}(\boldsymbol{Q}(0))\|_F.$$

Considering that

$$\|\hat{\varphi}(\boldsymbol{Q}(0))\|_F = \sqrt{\sum_{\iota=1}^{N}\operatorname{tr}(Q_\iota^*(0)Q_\iota(0))} = \|\boldsymbol{Q}(0)\|_1,$$

we have that

$$\|Q_\iota(k)\|_F \leq \|\Lambda\|_F\|\boldsymbol{Q}(0)\|_1,\ \forall\iota \in \mathcal{N}.$$

Thus,

$$\mathbf{E}[\|x(k)\|^2] \leq n\sum_{\iota=1}^{N}\|Q_\iota(k)\|_F \leq nN\,\|\Lambda\|_F\,\|\boldsymbol{Q}(0)\|_1.$$

Since (20) holds for any equivalent norm, and having

$$\|\boldsymbol{Q}(0)\|_1 = \sum_{\iota=1}^{N}\|Q_\iota(0)\| = \sum_{\iota=1}^{N}\|\mathbf{E}[x(0)x^*(0)\mathbf{1}_{\{\theta(0)=\iota\}}]\| \leq \sum_{\iota=1}^{N}\mathbf{E}[\|x_0\|^2\mathbf{1}_{\{\theta(0)=\iota\}}] = \|x_0\|_2^2,$$

we can finally write

$$\mathbf{E}[\|x(k)\|^2] \leq nN\beta'\zeta^k\|x_0\|_2^2 = \beta\zeta^k\|x_0\|_2^2$$

By Proposition 8 in the noiseless case (where the second and third summations in the expression of $Q_j(k+1)$ are equal to zero) and (16), it follows that, $\forall k \geq k' \in \mathbb{N}_0$,

$$\left\|\left(\prod_{i=0}^{k-1}A^*_{\theta(i)}\right)^*x(0)\right\|_2^2 \leq \beta\zeta^k\,\|x_0\|_2^2, \quad (21)$$

$$\beta = nN\beta',\quad \beta' = \zeta^{-k'}\sup_{\Pi\in\Pi_j[\mathcal{A}_L],\,0\leq j\leq k'}\|\Pi\|,\quad \beta' \geq 1, \quad (22)$$

$$\zeta \in (\hat{\rho}(\mathcal{A}_L), 1). \quad (23)$$

Following the same steps for the *second term* of the right hand side of (19), we obtain $\forall i\in\mathbb{N}_{k-1}$

$$\left\|\left(\prod_{j=i+1}^{k-1}A^*_{\theta(j)}\right)^*G_{\theta(i)}w(i)\right\|_2^2 \leq nN\beta'\zeta^{k-i-1}\|\boldsymbol{G}\boldsymbol{W}(i)\boldsymbol{G}^*\|_1,$$

with β' and ζ as in (22) and (23), respectively. Since

$$\|\boldsymbol{W}(i)\|_1 = \sum_{\iota=1}^{N}\|W_\iota(i)\| \leq \sum_{\iota=1}^{N}\mathbf{E}[\|w(i)\|^2\mathbf{1}_{\{\theta(i)=\iota\}}] = \mathbf{E}[\|w(i)\|^2] = \|w(i)\|_2^2,$$

we have that

$$\left\|\left(\prod_{j=i+1}^{k-1}A^*_{\theta(j)}\right)^*G_{\theta(i)}w(i)\right\|_2^2 \leq nN\beta'\zeta^{k-i-1}\|\boldsymbol{G}\|^2_{\max}\|w(i)\|_2^2. \quad (24)$$

From here on, our proof of necessity follows the steps of the proof provided in (Costa et al., 2005, Theorem 3.34) for time-homogeneous MJLS with bounded process noise. Applying the bounds obtained in (21) and (24), we can state that there $\exists\zeta\in(\hat{\rho}(\mathcal{A}_L),1)$ and $\beta'\geq 1$ such that

$$\|x(k)\|_2 \leq \sum_{i=0}^{k}\zeta_{k-i}\beta_i,$$

where

$$\zeta_{k-i} \triangleq (\sqrt{\zeta})^{(k-i)},\quad \beta_0 \triangleq (\sqrt{nN\beta'})\|x(0)\|_2,$$
$$\beta_i \triangleq (\sqrt{nN\beta'})\|\boldsymbol{G}\|_{\max}\,\|w(i-1)\|_2,\quad i\geq 1.$$

Let us set $a \triangleq (\zeta_0, \zeta_1, \dots)$ and $b \triangleq (\beta_0, \beta_1, \dots)$. Since $a\in\ell_1$ (i.e. $\sum_{i=0}^{\infty}|\zeta_i| < \infty$) and $b \in \ell_2$ (that is, $\sum_{i=0}^{\infty}|\beta_i|^2 < \infty$), it follows that the convolution $c \triangleq a * b = (c_0, c_1, \dots)$,

$c_k \triangleq \sum_{i=0}^{k} \zeta_{k-i}\beta_i$, lies itself in ℓ_2 with $\|c\|_2 \le \|a\|_1\|b\|_2$ (Costa et al., 2005, p. 56). Hence,

$$\|x\|_2 = \sqrt{\sum\nolimits_{k=0}^{\infty} \mathbf{E}[\|x(k)\|^2]} \le \sqrt{\sum\nolimits_{k=0}^{\infty} c_i^2} = \|c\|_2 < \infty.$$

This concludes the first part of the proof.

Let us prove sufficiency, that is $x = \{x(k); k \in \mathbb{N}_0\} \in \mathcal{C}^n$ $\forall w \in \mathcal{C}^r$, $x_0 \in \mathcal{C}_0^n$, $\theta_0 \in \Theta_0 \Rightarrow \hat{\rho}(\mathcal{A}_L) < 1$. By hypothesis,

$$\|x\|_2^2 = \sum\nolimits_{k=0}^{\infty} \mathbf{E}[\|x(k)\|^2] < \infty,\ \forall w \in \mathcal{C}^r,\ x_0 \in \mathcal{C}_0^n,\ \theta_0 \in \Theta_0.$$

$\mathbf{E}[\|x(k)\|^2] = \mathbf{E}[\mathrm{tr}(x(k)x^*(k))] = \mathrm{tr}(\mathcal{Q}(k)) \ge 0$, implies that $\|x\|_2^2 = \mathrm{tr}(\mathcal{Q}(k)) < \infty$, $\lim_{k\to\infty} \mathrm{tr}(\mathcal{Q}(k)) = 0$, Accordingly, as stated in (Costa et al., 2005, p. 44, within the proof of Proposition 3.24), this implies that

$$\lim_{k\to\infty} \mathcal{Q}(k) = 0, \quad \forall w \in \mathcal{C}^r,\ x_0 \in \mathcal{C}_0^n,\ \theta_0 \in \Theta_0.$$

Since the last statement holds for every $w \in \mathcal{C}^r$, we can make $w(k) = 0$, $\forall k \in \mathbb{N}_0$ in (3). From (12) we have that

$$\lim_{k\to\infty} \sum\nolimits_{i=1}^{n} Q_i(k) = 0, \quad Q_i(k) \in \mathbb{F}_+^{n\times n}.$$

Thus, from (8) it follows that $\lim_{k\to\infty} \boldsymbol{Q}(k) = 0$.

Since the mapping $\hat{\varphi}$ is uniform homeomorphic, also

$$\lim_{k\to\infty} \hat{\varphi}(\boldsymbol{Q}(k)) = 0$$

Applying (16), which holds when $w(k) = 0$, $\forall k \in \mathbb{N}_0$ in (3), we obtain

$$\lim_{k\to\infty} \left(\prod\nolimits_{i=0}^{k-1} \Lambda^*(i)\right)^{*} \hat{\varphi}(\boldsymbol{Q}(0)) = 0$$

From Proposition 4 and Remark 11, this last statement is true for every $Q(0)$ if and only if $\hat{\rho}(\mathrm{conv}\,\mathcal{A}_L) < 1$. From Proposition 3 follows the thesis. □

6. ILLUSTRATIVE EXAMPLE

In order to show that having the spectral radius smaller than one for each matrix Λ_i, $i \in \mathcal{N}$, is not enough to ensure the robust stability of the PTI system, let us consider the MJLS $(\mathcal{S})$ with $N = 3$ operational modes, where the state matrices associated with the operational modes are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.48 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.3 & 0.13 \\ 0.16 & 1.14 \end{bmatrix},$$

with $G_1 = 2\mathbb{I}_2$, $G_2 = 1.5\mathbb{I}_2$, $G_3 = \mathbb{I}_2$, $w(k) \in [-1, 1]^2 \subset \mathbb{R}^2$.

The time-varying probability matrix $P(k)$ is uncertain and belongs to a polytope with $L = 2$ vertices

$$P_1 = \begin{bmatrix} 0 & 0.35 & 0.65 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}.$$

Any probability matrix within a polytope is defined by

$$P(k) = \lambda(k)P_1 + (1 - \lambda(k))P_2, \quad 0 \le \lambda(k) \le 1.$$

Let us consider, e.g., also the matrix $P' = 0.5\,P_1 + 0.5\,P_2$. The spectral radii ρ of the matrices Λ are:

$$\rho(\Lambda_1) = 0.901601,\ \rho(\Lambda_2) = 0.905686,\ \rho(\Lambda') = 0.937965.$$

Thus, the time-homogeneous MJLS with TPM P_1, P_2 and P' are robustly (mean square) stable (Costa et al. (2005)). However, the PTI system having this TPMs is not robustly (mean square) stable, because the JSR, calculated with the JSR toolbox (Vankeerberghen et al. (2014)), is

$$\hat{\rho}(\mathcal{A}_L) = [\hat{\rho}_{min}(\mathcal{A}_L), \hat{\rho}_{\max}(\mathcal{A}_L)] = [1.024442, 1.031096]$$

This shows us that perturbations on transition probability matrix P can make a stable MJLS system unstable.

To present this result visually, we report one possible dynamical behavior of the system. For $x_0 = [100; 85]$ and the initial probability distribution $p_0 = [0.33, 0.34, 0.33]$, we have obtained the following system trajectories.

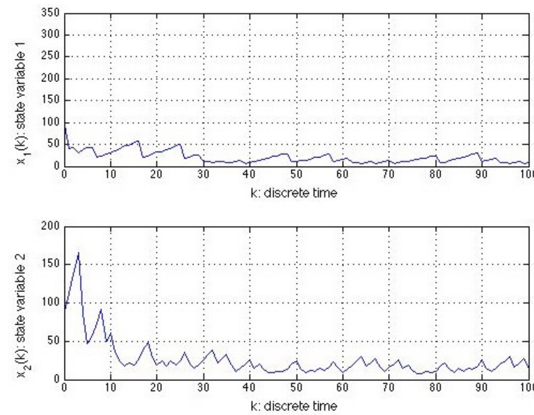

Fig. 1. A possible trajectory of $x(k)$ when TPM is P_1.

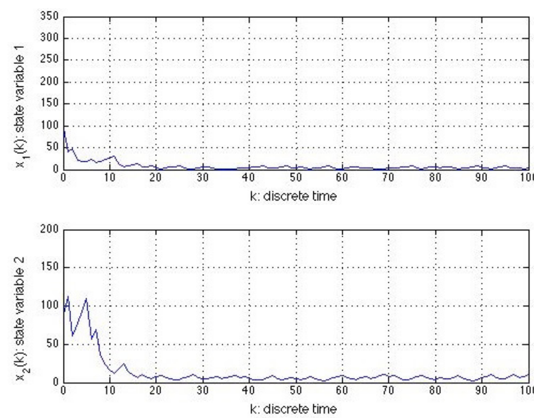

Fig. 2. A possible trajectory of of $x(k)$ when TPM is P_2.

Figure 1 shows us a trajectory of the $x(k)$ having only the time-homogeneous TPM P_1, while Figure 2 presents a system trajectory, when TPM used is always P_2.

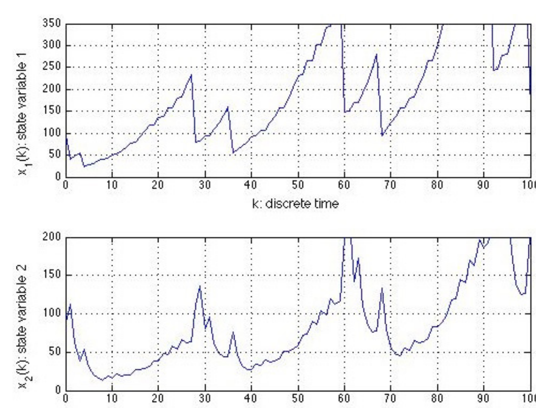

Fig. 3. A trajectory of $x(k)$ [$P(k)$ switching within P_1&P_2]

Figure 3 reveals a trajectory of the system state vector when the TPM is time-inhomogeneous and is switching within the polytope defined by vertices P_1 and P_2, evincing instability of the system.

However, by shrinking the polytope defining the uncertainty of the TPM to e.g. the new vertices

$$\bar{P}_1 = 0.8P_1 + 0.2P_2, \quad \bar{P}_2 = 0.2P_1 + 0.8P_2,$$

the corresponding time-inhomogeneous system is robustly (mean square) stable, because the joint spectral radius is

$$\hat{\rho}(\bar{\mathcal{A}}_L) = [\hat{\rho}_{min}(\bar{\mathcal{A}}_L), \hat{\rho}_{\max}(\bar{\mathcal{A}}_L)] = [0.971756, 0.972553].$$

Figure 4 reveals a trajectory of the system state vector when the TPM is time-inhomogeneous and is switching within the polytope defined by vertices $\bar{P}_1$ and $\bar{P}_2$, evincing robust stability of the system.

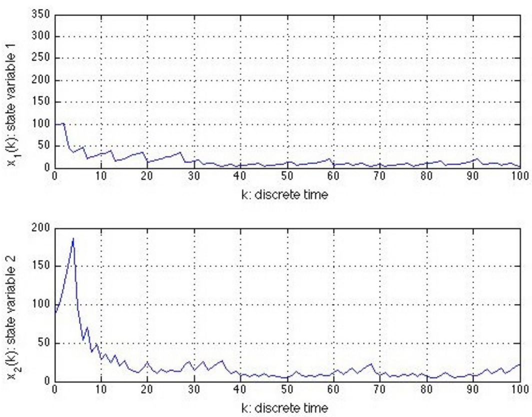

Fig. 4. A trajectory of $x(k)$ [$P(k)$ switching within $\bar{P}_1$&$\bar{P}_2$]

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