This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

Optimal output-feedback control over Markov wireless communication channels

Anastasia Impicciatore, Yuriy Zacchia Lun, Pierdomenico Pepe, and Alessandro D'Innocenzo

***Abstract*— The communication links connecting components of wireless control systems may be affected by packet losses due to time-varying fading and interference. We consider a wireless control network (WCN) with double-sided packet-losses: on the sensor–controller link (sensing link) and controller–actuator link (actuation link). We model the sensing and actuation links as finite-state Markov channels (FSMCs). One time-step delay affects the actuation link mode observation, while the sensing link mode observation is not affected by any delay. In this paper, we solve, as our main contribution, the optimal output-feedback control problem in this FSMC setting (under a TCP-like communication scheme) using two different state estimation techniques: Luenberger observer and current estimator, comparing the two methodologies and deriving a separation principle for both the cases. We also derive detectability conditions guaranteeing the existence of an optimal observer, either Luenberger or current.**

***Index Terms*— Wireless control networks, finite-state Markov channels, separation principle.**

I. Introduction

Wireless control networks (WCNs) consist of computational units, actuators, and sensors connected via wireless communication links that may be affected by packet losses. In wireless control systems literature, the packet dropouts have been modeled either as deterministic (in terms of time averages or worst case bounds on the number of consecutive packet losses, see, e.g., [1]–[3]) or stochastic phenomena. In the stochastic framework, many works in the literature assume memoryless packet drops, and thus dropouts are realizations of a Bernoulli process (see [4]–[7]). Other works consider more general correlated (bursty) packet losses and use a transition probability matrix (TPM) of a finite-state stationary Markov chain (see, e.g., [8] and references therein) to describe the stochastic process governing packet dropouts (see [4], [9], [10]). In these works, WCNs with missing packets are modeled via time-homogeneous Markov jump linear systems (MJLSs, [11]). Double-sided packet losses have been already investigated for instance in [3], [12], with arbitrary packet loss process [3], or Markovian [12]. These works summarize the packet losses on both links. The significant difficulty of this setting arises from a combined effect of two link packet losses possibly resulting in long periods in which the controller and actuator cannot simultaneously receive new data (see also Remark 1). However, a simple Markov chain model for packet losses on wireless channels used in WCNs literature is not exhaustive since the occurrence of packet losses also depends on the operational mode of the communication channel [8]. The finite-state stationary Markov channel (FSMC) model approximates the channel mode transitions through a Markov chain and incorporates a specific packet error distribution information into each mode. FSMC is an essential model because wireless communication system designers traditionally use this mathematical abstraction of the wireless channel for modeling error bursts in fading channels to analyze and improve performance measures in the physical or media access control layers. Moreover, several receivers' channel state estimation and decoding algorithms rely upon FSMC models [8]. Bursts of packet losses cannot be modeled by Bernoulli processes, which is the main limit of the output-feedback control (OFC) strategy based on Bernoulli channel. Indeed, the Bernoullian model is less accurate than the FSMC model, and thus bursts of packet losses may cause unstable behavior without the possibility of recovery, as illustrated in Section VIII. Thus, the existing stabilizability and detectability notions [4] are not suitable for the general FSMC scenario (see Remark 14). This work overcomes this limitation by solving the OFC problem over FSMCs and providing novel stabilizability and detectability conditions. The investigated infrastructure relies on a TCP-like architecture [4], implying that the communication between the controller and the actuators is characterized by acknowledgement (ack) messages. This paper generalizes the results in [4] to the FSMC setting also proving that the fundamental separation principle still remains valid when ack messages deliver the state of the channel and outcome of related transmission. Ack messages are crucial here because without them the separation between estimation and control is impossible even in Bernoullian setting. Concerning the transmission on the actuation link (AL), the controller is the transmitter: specifically, the transmitter cannot know the outcome of the transmission before sending the message. This is the reason why the controller receives the ack message, as well as the current mode of the channel only after a time-step delay [13], while this delay does not affect the sensing link (SL). In modern communication systems the channel state estimation is always performed through the receiver. Therefore, on the SL, the controller (i.e., the receiver) is able to know the outcome of the transmission and the Markov mode of the channel. The OFC for MJLSs has been investigated in [11] and [14], with the same Markov chain driving both the

The authors are with the Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila, L'Aquila 67100, Italy (e-mail: *anastasia.impicciatore@graduate.univaq.it*; *yuriy.zacchialun@univaq.it*; *pierdomenico.pepe@univaq.it*; *alessandro.dinnocenzo@univaq.it*)

dynamics of the plant and the one of the observer without any delay in the channel mode observation. Optimal linear quadratic regulation [15] with one time-step delay on AL mode has been investigated in [10]. In [16], the Kalman filter (KF) is adopted for a single simplified Gilbert channel modeled by a Markov chain with two states. This result cannot be applied to general Markov channel scenarios that require $2N$ modes with $N > 2$: N channel states result, e.g., from the signal-to-interference-plus-noise-ratio (SINR) partitioning, and each state is associated with a binary symmetric channel, see [8]. Thus, $2N$ modes derive from the general Markov channel mathematical model. Other estimation techniques are $\mathcal{H}_2$ and $\mathcal{H}_\infty$ estimation: in [9], sub-optimal filters are obtained for the case of cluster availability of the operational modes. It is well known that for the case in which the information on the output of the system and on the Markov chain are available at each time-step, the best linear estimator of the state is the KF (see [16] and [11, Remark 5.2]). An offline computation of the KF is inadvisable [17], as discussed more in detail in Section IV (Remark 9). On the other hand, an online computation of the KF requires a significant computational burden. For this reason, we consider a different class of estimators, for which we can pre-compute the filtering gains offline. We present two infinite horizon (IH) minimum mean square Markov jump filters [11, Ch.5.3]: the first one with a Luenberger observer (LO) and the second one using the current estimator (CE) [18, Ch. 8.2.4]. These estimators use different communication and computation timing sequences and offer different performance levels (see Section IV).

Preliminary versions: Preliminary parts of this work have been presented at the 58th IEEE Conference on Decision and Control [13] and at the 2021 American Control Conference [19]. Specifically, [13] has introduced the controllability notion over one step delayed AL mode observation, while [19] concerns the OFC with double-sided packet losses and detectability notions for the LO. The improvement with respect to [13] is the double-sided packet losses, while the novelty with respect to [19] is the introduction of the CE together with a comparison between the two methodologies. The CE provides better performance but it requires more restrictive constraints to be satisfied. Different computation timing sequences are used by the two estimators: the one concerning the CE presents more restrictive physical constraints (see Remark 13). The theoretical existence of these two estimators is a problem addressed using different detectability notions that have been introduced for the FSMC scenario and that are presented in this work with the aim of finding suitable conditions guaranteeing the existence of an observer (either Luenberger or current). Particularly, conditions guaranteeing the weakest detectability are necessary and sufficient, while requirements ensuring the strongest detectability are only sufficient. Moreover, we present the detailed proofs of the separation principle for LO and CE. Finally, we report a more general case study with respect to the one in [19], providing several propagation environments showing in which cases it is possible to conclude the existence of one of the two observers.

Paper contribution: The paper contributions are listed here.

(i) Firstly, the FSMC is introduced into wireless network control framework. The FSMC is widely used for analysis and design of telecommunication systems and allows for accurate modeling of errors and bursts of packet losses.

(ii) The communication timing, as well as computation and transmission delays are explicitly considered. This leads to two different estimation strategies: the LO and the CE, each one with its feasibility conditions.

(iii) The separation principle validity is proved for both the considered estimators in the general FSMC setting.

(iv) Four different detectability notions (presented from the weakest to the strongest one, see Remark 22) are introduced with the aim of providing a suitable theoretical basis for the formal description of the filtering problems. The aforementioned detectability notions are instrumental for the guarantees of the separation principle for the general FSMC scenario (see Remark 14).

(v) The presented results are illustrated in a case study concerning an inverted pendulum on a cart described in Section VIII.

Paper organization: The paper is organized as follows. Section II presents the wireless control network scenario and the information flow on AL and SL, respectively. Section III describes the optimal OFC problem in our setting. Estimation techniques are described and compared in Section IV and the corresponding observer stability analysis is provided in Section V (with the solutions of the filtering CAREs). Section VI states the separation principle derived for both the LO and the CE. Section VII presents the mode-independent output-feedback controller with suitable detectability conditions from the weakest to the strongest ones. A numerical case study is shown in Section VIII and some concluding remarks are reported in Section IX. Proofs of Lemmas and Theorems are reported in the appendix.

Notation and preliminaries: In the following, $\mathbb{N}$ denotes the set of natural numbers corresponding to the non-negative integers, $\mathbb{R}$ denotes the set of reals, while $\mathbb{F}$ indicates the set of either real or complex numbers. The absolute value of a number is denoted by $|\cdot|$. We recall that every finite-dimensional normed space over $\mathbb{F}$ is a Banach space [20] and denote the Banach space of all bounded linear operators of Banach space $\mathbb{X}$ into Banach space $\mathbb{Y}$, by $\mathbb{B}(\mathbb{X},\mathbb{Y})$. We set $\mathbb{B}(\mathbb{X},\mathbb{X}) \triangleq \mathbb{B}(\mathbb{X})$. $\mathbf{O}_n$ denotes the vector containing all zeros of length n. $\mathbb{I}_n$ indicates the identity matrix of size n, while $\mathbb{O}_n$ represents the matrix of zeros of size $n \times n$. The transposition is denoted by the apostrophe, the complex conjugation by an overbar, the conjugate transposition by superscript *. $\mathbb{F}_*^{n\times n}$ and $\mathbb{F}_+^{n\times n}$ represent the sets of Hermitian and positive semi-definite matrices, respectively. For any positive integers C, r, n, and m, we define the following sets: $\mathbb{H}^{Cr,n}$ is the set of all $\mathbf{K} = [K_m]_{m=1}^C$, K_m in $\mathbb{F}^{r\times n}$, $\mathbb{H}^{Cn,*}$ is the set of all $\mathbf{K} = [K_m]_{m=1}^C$, K_m in $\mathbb{F}_*^{n\times n}$, and $\mathbb{H}^{Cn,+}$ the set of all $\mathbf{K}$ in $\mathbb{H}^{Cn,*}$, with $K_m \in \mathbb{F}_+^{n\times n}$. We set $\mathbb{H}^{Cn} = \mathbb{H}^{Cn,n}$. We denote by $\rho(\cdot)$ the spectral radius of a square matrix (or a bounded linear operator), i.e., the largest absolute value of its eigenvalues, and by $\|\cdot\|$ either any vector norm or any matrix norm. We denote by $\otimes$ the Kronecker product defined in the usual way, see, e.g., [21], and $\oplus$ the direct sum. Notably, the direct sum of a sequence of square matrices $(\Phi_i)_{i=1}^C$ produces a

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

block diagonal matrix having its elements, Φ_i, on the main diagonal blocks. Then, $\mathrm{tr}(\cdot)$ indicates the trace of a square matrix. For two Hermitian matrices of the same dimensions, Φ_1 and Φ_2, $\Phi_1 \succeq \Phi_2$ (respectively $\Phi_1 \succ \Phi_2$) means that $\Phi_1 - \Phi_2$ is positive semi-definite (respectively positive definite). Finally, $\mathbb{E}(\cdot)$ stands for the mathematical expectation of the underlying scalar-valued random variable, and $\Re(\cdot)$ indicates the real part of the elements of a complex matrix.

Through this article we will extensively use the acronyms provided in the following: *MS* stands for mean-square, *MSS* stands for *MS* stable or *MS* stability, whose formal definition is provided in Section II. Moreover, *MSD* stands for *MS* detectability or *MS* detectable. The formal definition is provided in Section II.

II. PROBLEM FORMULATION

Consider the remote architecture depicted in Fig. 1. The discrete-time equivalent system is $\mathcal{G}$:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k^c + Gw_k, \\ y_k^s = Lx_k + Hw_k, \end{cases} \tag{1}$$

where the system state $x_k \in \mathbb{F}^{n_x}$ and the system output $y_k^s \in \mathbb{F}^{n_y}$, $k \in \mathbb{N}$, are obtained through an Analog to Digital Converter (A/D block in Fig. 1) with sampling period T. For $k \in \mathbb{N}$, $w_k \in \mathbb{R}^{n_w}$ is a sequence of i.i.d. Gaussian random variables (RVs) with zero mean. A, B, G, L, and H are system matrices of appropriate sizes. As in [4], we consider an unstable system state matrix A since otherwise a stabilizing output-feedback control would not be required. $\mathcal{G}$ is controlled remotely by a digital output-feedback controller, which receives the measurements y_k^s on the wireless SL and sends the control inputs over the wireless AL. The received digital control law $u_k^c \in \mathbb{F}^{n_u}$ is converted to an analog signal by a Digital to Analog Converter (D/A block in Fig. 1) based for instance on Zero-Order Hold (ZOH), so that the analog control input can be applied to the continuous-time process.

Remark 1: Fig. 1 reports the scheme of a WCN infrastructure with possible packet losses occurrence on both the SL and AL. The main challenge of this scenario arises from a combined effect of two link packet losses possibly resulting in long periods in which the controller and actuator cannot simultaneously receive new data. The scheme is a TCP-like communication [4] based on ack messages. Specifically, the controller receives the ack of the transmission on the connection actuators-controller (see Fig. 1). Packet losses over this connection are negligible since the probability of a packet loss for ack messages is very small in practical applications.

A. WIRELESS LINK

This section describes single-hop wireless communication links modeled by FSMCs. The sequence $\{\nu_k\}_{k\in\mathbb{N}}$ models the packet arrival process on the AL. The value of the RV ν_k is zero whenever the control packet is lost, and $\nu_k = 1$ if the control packet is correctly delivered, i.e., $\nu_k \in \mathbb{S}_\nu \triangleq \{0, 1\}$, for any $k \in \mathbb{N}$. Analogously, the sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ describes the packet arrival process on the wireless SL. Particularly, $\gamma_k = 0$ if the sensing packet is lost and $\gamma_k = 1$ if it is successfully delivered, i.e., for all $k \in \mathbb{N}$, $\gamma_k \in \mathbb{S}_\gamma \triangleq \{0, 1\}$. The processes ν_k and γ_k are collections of binary RVs and the probability of having a packet loss or a correct packet transmission over each link depends on its SINR. The SINR is determined by propagational environment and related physical phenomena [22]. SINR is a stochastic process and can be abstracted by a Markov chain. Each Markov mode is associated with a certain packet error probability (PEP). We consider the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in\mathbb{N}}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of (Borel) measurable events, $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ is the related filtration, and $\mathbb{P}$ is the probability measure. SL and AL modes are the output of the Markov chains $\eta : \mathbb{N} \times \Omega \to \mathbb{S}_\eta \subseteq \mathbb{N}$ and $\theta : \mathbb{N} \times \Omega \to \mathbb{S}_\theta \subseteq \mathbb{N}$, respectively. Indeed, the Markov modes of $\{\eta_k\}_{k\in\mathbb{N}}$ and $\{\theta_k\}_{k\in\mathbb{N}}$ belong to finite sets $\mathbb{S}_\eta = \{1, 2, \ldots, I\}$ and $\mathbb{S}_\theta = \{1, 2, \ldots, N\}$, respectively.

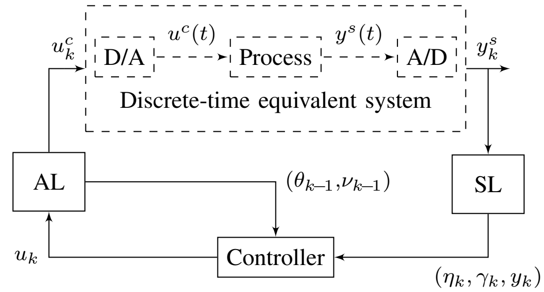


Fig. 1. Remote output-feedback architecture.

Remark 2: Previous works such as [4], [5] do not consider the communication channel mode, but actually the receiver has access to this information, by performing a channel state estimation [23]. The novelty of this paper lies within the OFC in the FSMC setting.

Moreover, the described Markov chains are characterized by time-invariant transition probability matrices (hereafter, TPMs) $P = [p_{ij}]_{i,j=1}^N$ (for $\{\theta_k\}$) and $Q = [q_{mn}]_{m,n=1}^I$ (for $\{\eta_k\}$), respectively. Each TPM may be obtained by integrating the joint probability density function of the SINR over two consecutive packet transmissions and over the desired regions [8], [22]. The TPM values may also be validated through the empirical data from a measurement campaign for calibrating the theoretical model parameters. The uncertainties in TPM values neglected in this work can then be addressed via a polytopic model (e.g., [24] and the references therein).

Remark 3: The network-induced communication delays due to multiple path routing and time-varying processing delays in relay nodes of multi-hop networks are not an issue for single-hop sensing and actuation links with scheduled medium access considered in this paper and extensively used in delay-sensitive control applications relying, e.g., on the low latency deterministic network mode of IEEE 802.15.4e.

The entries of TPMs P and Q are defined as

$$p_{ij} \triangleq \mathbb{P}(\theta_{k+1} = j \mid \theta_k = i), \quad q_{mn} \triangleq \mathbb{P}(\eta_{k+1} = n \mid \eta_k = m), \tag{2}$$

satisfying: $\sum_{j\in\mathbb{S}_\theta} p_{ij} = 1$, $\sum_{n\in\mathbb{S}_\eta} q_{mn} = 1$, $i \in \mathbb{S}_\theta$, $m \in \mathbb{S}_\eta$. Since the probability of a packet loss depends on the mode of the Markov chain, the values of ν_k and γ_k are either zero or one with certain probabilities depending on the current Markov mode.

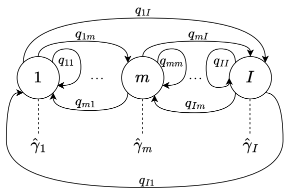


Fig. 2. FSMC model for SL: the Markov chain $\boldsymbol{\eta_k}$ represents the evolution of the channel, while successful packet delivery and PER come from $\hat{\boldsymbol{\gamma}}_{\boldsymbol{m}}, \boldsymbol{m} \in \mathbb{S}_{\boldsymbol{\eta}}$.

Remark 4: In this network scenario, up-link and down-link models are split up. This separation already exists in literature [3], [4]. However, unlike the previous literature, we explicitly consider the channel mode (see Remark 2) by providing two independent FSMCs for SL and AL, respectively.

1) Sensing FSMC: Let y_k denote the measurement received by the output-feedback controller at time $k \in \mathbb{N}$. The general model for the SL is $y_k = \gamma_k y_k^s$: the value of the RV γ_k when the current Markov mode is $\eta_k \in \mathbb{S}_\eta$ is a function of η_k, and, for notational convenience, we denote it as $\gamma_k = \gamma(\eta_k)$. The probability of having a successful packet delivery on the SL depends on the current Markov mode $\eta_k = m$, i.e.,

$$\hat{\gamma}_m \triangleq \mathbb{P}(\gamma_k = 1 \mid \eta_k = m), \ \ \mathbb{P}(\gamma_k = 0 \mid \eta_k = m) = 1 - \hat{\gamma}_m, \quad (3)$$

are the probability that the packet is successfully delivered at time $k \in \mathbb{N}$, and the likelihood of a packet loss occurrence conditioned to $\eta_k = m$, respectively. Fig. 2 provides a graphical representation of the FSMC model on the SL. A visual representation of the AL is similar, and thus omitted for brevity. Let $\pi_m(k)$ denote the probability $\mathbb{P}(\eta_k = m)$, for $m \in \mathbb{S}_\eta$, $k \in \mathbb{N}$. The variable $\pi_m(k)$ can also be written through the indicator function $\mathbf{1}_{\{\eta_k = m\}}$, as $\pi_m(k) = \mathbb{E}[\mathbf{1}_{\{\eta_k = m\}}]$, see [11]. We do set $\boldsymbol{\pi}(k) = [\pi_m(k)]_{m=1}^I$.
For what concerns the process $\{\gamma_k\}$, applying Bayes Law, the Markov property, and the independence between $\{\gamma_k\}$ and $\{\eta_k\}$, we obtain, for $m, n \in \mathbb{S}_\eta$, [19]

$$\mathbb{P}(\gamma_{k+1} = 1, \eta_{k+1} = n \mid \eta_k = m) = \hat{\gamma}_n q_{mn},$$
$$\mathbb{P}(\gamma_{k+1} = 0, \eta_{k+1} = n \mid \eta_k = m) = (1 - \hat{\gamma}_n) q_{mn}.$$

2) Actuation FSMC: In the SL, the controller is the receiver and has direct access to the channel information (see Remark 2). For the AL, the controller is the transmitter and can access the actuation channel information by an ack message, as the reader may notice in Fig. 1. Obviously, the ack message is received after the transmission, so there is a one time-step delay. Let $u_k \in \mathbb{F}^{n_u}$ denote the control law computed by the controller, and let u_k^c denote the digital control input received by the D/A block at time $k \in \mathbb{N}$. The general model for the AL is $u_k^c = \nu_k u_k$: the value of the RV ν_k when the current Markov mode is $\theta_k \in \mathbb{S}_\theta$ is a function of θ_k, and, for notational convenience, we denote it as $\nu_k = \nu(\theta_k)$.

The probability of the correct packet delivery on AL depends on the current mode of the AL, that is $\theta_k = i$, i.e.,

$$\hat{\nu}_i \triangleq \mathbb{P}(\nu_k = 1 \mid \theta_k = i), \ \ \mathbb{P}(\nu_k = 0 \mid \theta_k = i) = 1 - \hat{\nu}_i, \quad (4)$$

are the probability that the packet is correctly delivered at time $k \in \mathbb{N}$, and the likelihood that the control packet is lost conditioned to $\theta_k = i$, respectively. For $i \in \mathbb{S}_\theta$, $k \in \mathbb{N}$, the probability $\mathbb{P}(\theta_k = i)$ is denoted by $\varpi_i(k)$. For $\ell, i \in \mathbb{S}_\theta$, $k \in \mathbb{N}$, the joint probability of being in an augmented Markov state (θ_{k-1}, θ_k), $\mathbb{P}(\theta_{k-1} = \ell, \theta_k = i)$ is denoted by $\tilde{\varpi}_{\ell i}(k)$. Moreover, the quantity $\tilde{\varpi}_{\ell i}(k)$ may be written using the indicator function $\mathbf{1}_{\{\theta_{k-1} = \ell, \theta_k = i\}}$, as $\tilde{\varpi}_{\ell i}(k) = \mathbb{E}[\mathbf{1}_{\{\theta_{k-1} = \ell, \theta_k = i\}}]$ [11]. We do set $\tilde{\boldsymbol{\varpi}}(k) = [\tilde{\varpi}_{\ell i}(k)]_{\ell, i=1}^N$. The probability $\tilde{\varpi}_{\ell i}(k)$ evolves according to the following equations, for $\ell, i \in \mathbb{S}_\theta$, $k \in \mathbb{N}$ [13]:

$$\mathbb{P}(\theta_{k+1} = j, \theta_k = i \mid \theta_k \neq i, \theta_{k-1} = \ell) = 0,$$
$$\mathbb{P}(\theta_{k+1} = j, \theta_k = i \mid \theta_k = i, \theta_{k-1} = \ell) = p_{ij}.$$

Recalling that the availability of AL mode is affected by one time-step delay, that is θ_{k-1} (see Fig.1), the aggregated Markov state (θ_k, θ_{k-1}) is considered [13]. This memory introduced by the presented aggregation is fictitious: the aggregated Markov chain satisfies the Markov property of the memoryless chain $\{\theta_k\}$. Moreover, we can compute the probabilities of the joint process $(\nu_k, \theta_k, \theta_{k-1})$ as in [13].

3) The information set: The scenario depicted in Fig. 3 shows the information flow of actuation and sensing data between the plant and the controller, under TCP-like protocols, i.e., in the presence of ack messages [4]. Transmissions and computations do not happen instantly: as the reader may see in Fig. 3, actuation and sensing transmission time (δ_3 and δ_1, respectively) are greater than zero, as well as the control law computation time (denoted by δ_2) and the ack transmission time δ_4. Two different scenarios may arise: either the time interval δ_2 needed to the controller for the computations of estimation and control law is comparable to the sampling period T (this may happen when slow computers are used to control high-order systems) or the time needed for the estimation is very small compared to the sampling period [18]. The first case is depicted in Fig. 3-(a), where the computation time δ_2 is comparable to the sampling period T. The suitable estimation technique in this case is provided by the LO, that requires the measurements up through the previous time instant [18, Ch. 8]. By considering the delay δ_1 introduced by the sensing transmission, the controller owns the whole information necessary for the estimation needed in the computation of u_{k+1} exactly at $kT + \delta_1$. Formally, the information set available to the output-feedback controller for the computation of u_{k+1}, based on the LO is $\mathcal{F}_l^{k+1} = \{(u_t)_{t=0}^k, (y_t)_{t=0}^k, (\nu_t)_{t=0}^k, (\gamma_t)_{t=0}^k\}$. The information set $\mathcal{F}_l^{k+1}$ implies that in the Luenberger-based output-feedback u_{k+1} does not depend on the most recent observation [18, Ch. 8]. Thus, the estimate vector might not be as accurate as the one obtained with the most recent measurement. For high-order systems controlled by slow computers, or whenever the sampling periods are comparable to the computation time, the time interval between the observation instant and the validity time of the control output allows the computer to complete the calculations [18]. In many systems however, the computation time required to evaluate the estimation is quite short compared to the sampling period (see δ_5 in Fig. 3-(b)), and the delay of almost a cycle between the measurement and the proper time to apply the resulting control calculation represents an unnecessary waste. Therefore, the controller may exploit the current output measurement to obtain a more accurate state estimation. Fig. 3-(b) shows the time diagram of a two-step estimation algorithm: the first step predicts the state estimate based on the measurement from the previous time-step, while the following step corrects the predicted estimate by integrating the most recent measurement. The time needed to perform the last step (concerning the estimate correction and control law computations) denoted by δ_5, is contained in δ_2, and its brevity enables the control law transmission within the proper time window, coherently with the scenario described above of a controller with higher performance, [18, Ch. 8]. Notably, the current measurement is used within a different estimation technique (hereafter, current estimator or CE) that provides a more accurate estimated state vector based on the most recent output information. The information set used for computing u_k during δ_5, denoted by $\mathcal{F}_c^k$, collects the information received up through $kT+\delta_1$. Formally, $\mathcal{F}_c^k \triangleq \{(u_t)_{t=0}^{k-1}, (y_t)_{t=0}^{k}, (\nu_t)_{t=0}^{k-1}, (\gamma_t)_{t=0}^{k}\}$. We emphasize that the current control input based on $\mathcal{F}_c^k$ has access to the most recent observation. Exploiting this additional information considerably increases the performance resulting in lower estimation error cost, as explained more in detail in Sections IV and VIII.

Remark 5: The state-system model (1) does not explicitly account for the sensing and actuation delays (δ_1 and either $\delta_2+\delta_3$ or $\delta_5+\delta_3$ for the LO and CE, respectively) below one sampling period T. Completely neglecting these delays may reduce the system stability. However, for the state-space-based design, an actuation delay of a fraction of a sampling period corresponds to augmenting the system, while the sensing delay does not influence the sampled value [18, Ch. 4.3.4, 8.6]. Thus, without loss of generality, we consider the system matrices in (1) as augmented to account for the sub-sampling-period delays.

Remark 6: A natural alternative to the considered estimators is the mode-independent estimator based on KF described in [4] by Schenato et al., which does not require a channel state estimation and thus results in a less complex design. However, the estimator in [4] may fail to support a stable OFC over FSMCs, as discussed in Section VIII. The necessary condition for a stable mode-independent estimation and control over fading Markov channels is the system should behave well, i.e., it should be *Strong-MSD* and Strong-MS stabilizable, as detailed in Section VII.

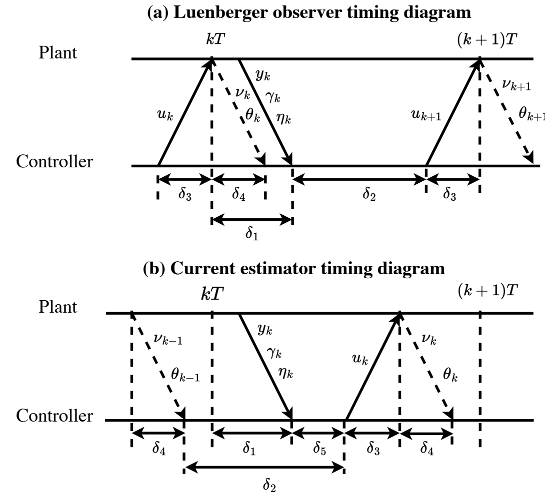
B. Wireless control network model

Given the system described by (1) and actuation and sensing FSMCs, the stochastic system describing the architecture in Fig. 1 can be written as follows

$$\begin{cases} x_{k+1} = Ax_k + \nu_k Bu_k + Gw_k, \\ y_k = \gamma_k Lx_k + \gamma_k Hw_k, \\ z_k = Cx_k + \nu_k Du_k, \end{cases} \tag{5}$$

with $z_k \in \mathbb{F}^{n_z}$ (needed to define the performance index of the optimal controller), C and D matrices of appropriate sizes.

Remark 7: Both ν_k and γ_k depend on the corresponding channel mode according to the FSMC model, i.e., $\gamma_k = \gamma(\eta_k)$ and $\nu_k = \nu(\theta_k)$, respectively (see Section II-A). Therefore, we refer to the system described by (5) as MJLS.



δ_1 Sensing transmission time (from sensors to controller)
δ_2 Computation time of the controller
δ_3 Actuation transmission time (from controller to actuators)
δ_4 Acknowledgment transmission time
δ_5 Time needed for estimate correction with the current output measurement

Fig. 3. Information flow timing between the plant and the controller used for the LO (a) and the CE (b).

We assume that noise sequence $\{w_k\}$ is independent of the initial state x_0 and the sequences $\{\nu_k\}$ and $\{\gamma_k\}$. Moreover,

$$\mathbb{E}[w_k] = \mathbf{O}_{n_w}, \quad \mathbb{E}[w_k w_k^*] = \mathbb{I}_{n_w}, \quad \mathbb{E}[w_k w_l^*] = \mathbb{O}_{n_w}, \tag{6}$$

$\forall k, l \in \mathbb{N}, k \neq l$, see also [11]. We assume, without loss of generality, that the system matrices are constant matrices of appropriate sizes [11, Sec. 5.2], such that

$$GH^* = 0, \;\; HH^* \succ 0, \;\; C^*D = 0, \;\; D^*D \succ 0. \tag{7}$$

Similarly to [11, Sec. 5.3], we make the following technical assumptions (with $k \in \mathbb{N}$):

$a.1)$ initial conditions x_0, θ_0, and η_0 are independent RVs,

$a.2)$ white noise sequence $\{w_k\}$ is independent of initial conditions (x_0, ν_0, γ_0) and of processes $\{\nu(\theta_k)\}$ and $\{\gamma(\eta_k)\}$, for any k,

$a.3)$ Markov chains $\{\theta_k\}$, $\{\eta_k\}$ and the noise sequence $\{w_k\}$ are independent,

$a.4)$ Markov chains $\{\theta_k\}$ and $\{\eta_k\}$ are ergodic, with steady-state probability distributions $\tilde{\varpi}_{\ell i}^{\infty} \triangleq \lim_{k\to\infty} \tilde{\varpi}_{\ell i}(k)$, $\varpi_i^{\infty} \triangleq \lim_{k\to\infty} \varpi_i(k)$, and $\pi_m^{\infty} \triangleq \lim_{k\to\infty} \pi_m(k)$, $\ell, i \in \mathbb{S}_\theta$ and $m \in \mathbb{S}_\eta$. We set $\tilde{\boldsymbol{\varpi}}^{\infty} = [\tilde{\varpi}_{\ell i}^{\infty}]_{\ell,i=1}^{N}$ and $\boldsymbol{\pi}^{\infty} = [\pi_m^{\infty}]_{m=1}^{I}$.

This paper aims to solve the OFC problem over FSMCs with two different estimation techniques guaranteeing the IH convergence of the state in *MS*. This property is known as *MSS* [11, Definition 3.8, pp. 36–37] that we present as follows.

Definition 1: The MJLS described by (5) is *MSS* if there exist equilibrium points $\hat{\mu}$ and $\widehat{Q}$ (independent from initial conditions) such that, for any initial condition (x_0, ν_0, γ_0), the following equalities hold: $\lim_{k\to\infty} \|\mathbb{E}(x_k) - \hat{\mu}\| = 0$, $\lim_{k\to\infty} \|\mathbb{E}(x_k x_k^*) - \widehat{Q}\| = 0$.

III. Output-feedback controller

This section shows two alternative OFC systems for the problem formalized in Section II.

A. Control synthesis based on Luenberger observer

Consider the scenario in Fig. 3-(a) and the related information set $\mathcal{F}_l^k$, $k \in \mathbb{N}$. The optimal LO-based Markov jump OFC system relying on $\mathcal{F}_l^k$ for the synthesis of u_k is

$$\mathcal{G}_l : \begin{cases} \check{x}_{k+1} = \check{A}(\nu_k, \theta_{k-1}, \gamma_k, \eta_k)\check{x}_k + \check{B}(\eta_k)y_k, \\ u_k = \check{F}(\theta_{k-1})\check{x}_k, \end{cases} \tag{8}$$

with $\check{x}_k$ being the estimated state obtained by the LO. The controller $\mathcal{G}_l$ (with optimal matrices $\check{A}(\nu_k, \theta_{k-1}, \gamma_k, \eta_k)$, $\check{B}(\eta_k)$, and $\check{F}(\theta_{k-1})$ to be found) should guarantee *MSS* of the closed-loop system (see Definition 1). The sequences of matrices $\check{\mathbf{F}} = [\check{F}(\ell)]_{\ell=1}^N$ and $\check{\mathbf{B}} = [\check{B}(n)]_{n=1}^I$ are the solutions of the optimal control and of the optimal filtering problem, respectively.

B. Output-feedback controller with current estimator

Consider the scenario in Fig. 3-(b) and the related information set $\mathcal{F}_c^k$. The optimal CE-based Markov jump OFC system relying on $\mathcal{F}_c^k$ for the synthesis of u_k is

$$\mathcal{G}_c : \begin{cases} \tilde{x}_{k+1} = \widehat{A}(\gamma_k, \eta_k)\tilde{x}_k + \widehat{B}(\eta_k)y_k + \widehat{C}(\nu_k, \theta_{k-1})\hat{x}_k, \\ \hat{x}_{k+1} = \tilde{x}_{k+1} + \widehat{D}(\eta_{k+1})\,[y_{k+1} - \gamma_{k+1}L\tilde{x}_{k+1}], \\ u_k = \widehat{F}(\theta_{k-1})\hat{x}_k, \end{cases} \tag{9}$$

with $\tilde{x}_k$ and $\hat{x}_k$, prediction state and correction state at time $k \in \mathbb{N}$, respectively, obtained using the CE. The controller $\mathcal{G}_c$ (with optimal matrices $\widehat{A}(\gamma_k, \eta_k)$, $\widehat{B}(\eta_k)$, $\widehat{C}(\nu_k, \theta_{k-1})$, $\widehat{D}(\eta_{k+1})$, and $\widehat{F}(\theta_{k-1})$ to be found) should guarantee the *MSS* of the closed-loop system (see Definition 1). The sequences of matrices $\widehat{\mathbf{F}} = [\widehat{F}(\ell)]_{\ell=1}^N$ and $\widehat{\mathbf{D}} = [\widehat{D}(n)]_{n=1}^I$ are the solutions of optimal control and filtering problem, respectively.

Remark 8: Both $\mathcal{G}_l$ and $\mathcal{G}_c$ should achieve the *MSS* of the closed-loop system. The CE provides a valid alternative to the LO and the proper control strategy should be chosen according to the calculating capacity of the controller. When the computation time δ_5 (see Fig. 3-(b)) required for the correction of the predicted estimate is under a certain threshold, the suitable controller is $\mathcal{G}_c$, otherwise $\mathcal{G}_l$ should be preferred, see also Remark 9.

C. The linear quadratic regulator

The necessary condition for an optimal IH solution of the wireless control problem is the *MS* stabilizability with delay.

Definition 2 (MS stabilizability with delay): The system (5) is *MS* stabilizable with one time-step delayed AL mode observation if, for any initial condition (x_0, θ_0), and for each mode $\ell \in \mathbb{S}_\theta$, there exists a mode-dependent gain F_ℓ, such that $u_k = F_{\theta_{k-1}} x_k$ is the *MS* stabilizing state-feedback for (5). Let $F_\ell \in \mathbb{F}^{n_u \times n_x}$, $\ell \in \mathbb{S}_\theta$, denote the optimal mode-dependent control gain with one time-step delayed operational mode observation of the AL (see [13] for the solution of the IH optimal control problem and [10] for a more general result). For any $\mathbf{X} = [X_l]_{l=1}^N \in \mathbb{H}^{Nn_x,+}$, $l \in \mathbb{S}_\theta$, let us define $\mathcal{A}_l(\mathbf{X})$ and $\mathcal{C}_l(\mathbf{X})$ as follows: $\mathcal{A}_l(\mathbf{X}) \triangleq A^*(\sum_{i=1}^N p_{li}X_i)A + C^*C$, $\mathcal{C}_l(\mathbf{X}) \triangleq A^*(\sum_{i=1}^N p_{li}\hat{\nu}_i X_i)B$. Let us also define $\mathcal{B}_l(\mathbf{X})$ as $\mathcal{B}_l(\mathbf{X}) \triangleq \sum_{i=1}^N p_{li}\hat{\nu}_i(B^*X_iB + D^*D)$ and $\mathcal{X}_l(\mathbf{X})$ as $\mathcal{X}_l(\mathbf{X}) \triangleq \mathcal{A}_l(\mathbf{X}) - \mathcal{C}_l(\mathbf{X})\mathcal{B}_l^{-1}(\mathbf{X})\mathcal{C}_l^*(\mathbf{X})$. For $l \in \mathbb{S}_\theta$, the set of equations $X_l = \mathcal{X}_l(\mathbf{X})$ is the set of control coupled algebraic Riccati equations (hereafter, control CAREs). The necessary condition for the existence of the *MS* stabilizing solution $\tilde{\mathbf{X}} \in \mathbb{H}^{Nn_x,+}$ of the control CAREs is the *MS* stabilizability with delay of system (5) (see Definition 2). If $\tilde{\mathbf{X}} \in \mathbb{H}^{Nn_x,+}$ is the *MS* stabilizing solution of the control CAREs, then the state-feedback control input $F_{\theta_{k-1}}x_k$ stabilizes the system, with one time-step delay in the observation of the AL mode in the *MS* sense (see [13]). The optimal control problem solution is obtained by using the LMI approach [10]. The optimized performance index is given by $J_h = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\Big[\sum_{k=0}^t (z_k z_k^*)\,|\mathcal{F}_h^k\Big]$, with z_k in (5), $h = l$ for the LO and $h = c$ for the CE. The performance index achieved by the optimal control law is $J_h^* = \sum_{i=1}^N \varpi_i^\infty \mathrm{tr}\,(G^*X_iG)$.

IV. Estimation techniques

The output-feedback controllers introduced in Section III rely either on the LO ($\mathcal{G}_l$) or on the CE ($\mathcal{G}_c$). The aim of the control law is ensuring the MSS of the closed-loop system. The aim of each estimator is ensuring *MSS* of the estimation error dynamical system associated with the estimation technique.

Definition 3: The MJLS (5) is *MSD* if there exists an estimator such that the corresponding estimation error system is *MSS*.

Remark 9: For the case in which the information on the output of the system and on the Markov chain are available at each time-step, the best linear estimator of $x(k)$ is the KF (see [11, Remark 5.2]). In offline computations of the KF, the solutions of the difference Riccati equations and of the time-varying Kalman gain are sample-path dependent, and the number of sample paths grows exponentially in time. Thus, KF offline implementation is inadvisable here [17]. On the other hand, an online implementation of the KF requires online matrix inversions which might have a heavy computational burden. Therefore, this work takes into account a different class of estimators with filtering gains pre-computed offline. This avoids online matrix inversions and reduces the computational burden.

A. The Markovian Luenberger observer

This subsection briefly recalls the Markovian LO presented in [19], given by

$$\check{\mathcal{G}} : \begin{cases} \check{x}_{k+1} = A\check{x}_k + \nu_k B u_k - \check{M}_{\eta_k}(y_k - \gamma_k L\check{x}_k), \\ u_k = F_{\theta_{k-1}}\check{x}_k, \\ \check{x}(0) = \check{x}_0, \end{cases} \tag{10}$$

with $\check{M}_m$, $m \in \mathbb{S}_\eta$, mode-dependent filtering gain obtained as solution of the Luenberger filtering problem, which relies on the information set $\mathcal{F}_l^k$. Note that when the controller makes the computations for $\check{x}_{k+1}$, it knows whether the packets containing the control law u_k and the measurement y_k have been

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

received or not. Indeed, this information is contained in $\mathcal{F}_l^{k+1}$, which is exploited for computing the proper control input to apply at time $k+1$, that is $u_{k+1}=F_{\theta_k}\check{x}_{k+1}$. Let us define the LO estimation error at time-step $k\in\mathbb{N}$ as $\check{e}_k\triangleq x_k-\check{x}_k$. The error dynamics are derived as follows:

$$\check{e}_{k+1}=\left(A+\gamma_k\check{M}_{\eta_k}L\right)\check{e}_k+\left(G+\gamma_k\check{M}_{\eta_k}H\right)w_k. \quad (11)$$

B. The Markovian current estimator

The CE [18, Ch. 8] over the FSMC results in the following MJLS,

$$\widehat{\mathcal{G}}:\begin{cases}\hat{x}_{k+1}=\tilde{x}_{k+1}-\widehat{M}_{\eta_{k+1}}\left[y_{k+1}-\gamma_{k+1}L\tilde{x}_{k+1}\right],\\ y_{k+1}=\gamma_{k+1}Lx_{k+1}+\gamma_{k+1}Hw_{k+1},\\ u_k=F_{\theta_{k-1}}\hat{x}_k,\end{cases} \quad (12)$$

with $\widehat{M}_m$, $m\in\mathbb{S}_\eta$, mode-dependent filtering gain obtained by solving the CE problem that relies on the information set $\mathcal{F}_c^k$ [18]. The variables $\tilde{x}_k$ and $\hat{x}_k$ are the predicted and the estimated state vectors at time-step $k\in\mathbb{N}$, respectively. The CE is a two-step estimation algorithm: the first step computes the prediction $\tilde{x}_{k+1}=A\hat{x}_k+\nu_kBu_k$ based on the measurement from the previous time-step, while the following step corrects the predicted estimate by integrating the most recent measurement. The estimated state vector resulting from this correction with y_{k+1} is $\hat{x}_{k+1}$.
Define the prediction error at time-step $k\in\mathbb{N}$ as $e_k\triangleq x_k-\tilde{x}_k$. The resulting estimated state Markov jump system is

$$\hat{x}_{k+1}=\tilde{x}_{k+1}-\gamma_{k+1}\widehat{M}_{\eta_{k+1}}Le_{k+1}-\gamma_{k+1}\widehat{M}_{\eta_{k+1}}Hw_{k+1}. \quad (13)$$

Remark 10: At time-step $k+1$, the predicted state $\tilde{x}_{k+1}$ is corrected exploiting the prediction error e_{k+1}, through the most recent output measurement.
By substituting $\hat{x}_k$, obtained from (13), in the prediction, the expression of $\tilde{x}_{k+1}$ depends on the prediction error, as follows,

$$\tilde{x}_{k+1}=A\tilde{x}_k+\nu_kBu_k-\gamma_kA\widehat{M}_{\eta_k}Le_k-\gamma_kA\widehat{M}_{\eta_k}Hw_k. \quad (14)$$

Therefore, the prediction error MJLS is given by

$$e_{k+1}=\left(A+\gamma_kA\widehat{M}_{\eta_k}L\right)e_k+\left(G+\gamma_kA\widehat{M}_{\eta_k}H\right)w_k. \quad (15)$$

Define the CE estimation error as $\hat{e}_k\triangleq x_k-\hat{x}_k$. Consequently,

$$\hat{e}_{k+1}=e_{k+1}+\gamma_{k+1}\widehat{M}_{\eta_{k+1}}Le_{k+1}+\gamma_{k+1}\widehat{M}_{\eta_{k+1}}Hw_{k+1}. \quad (16)$$

Remark 11: In the LO, the estimation error coincides with the prediction error. In the CE, when the prediction error e_k converges to zero, by (16), the estimation error goes to zero. Thus, (16) and (15) are equivalent at the steady-state.

Remark 12: Neither the control input nor the Markov chain $\{\theta_k\}$ are involved in the MJLSs (11) and (15). This implies that the optimal mode-dependent LO gain $\check{M}_m$ and the CE gain $\widehat{M}_m$, $m\in\mathbb{S}_\eta$, can be designed independently from the optimal mode-dependent control gain F_ℓ, $\ell\in\mathbb{S}_\theta$.

C. Computation time

It is well known that the total number of *floating-point operations* or *flops* to carry out the presented estimation algorithms may provide a rough estimate of the computation time [25]. Given the state estimate vector, the number of flops needed for the evaluation of the control law is $O(n_un_x)$. Moreover, the computational complexity of both the Luenberger and the current state estimation numerical algorithms is the same: $O(n_x^2+n_xn_u+n_y+n_xn_y)$. The physical constraint for estimator implementation is obtained comparing δ_2 (the time needed for all the computations leading to the control law) and the sampling time T. If the condition $\delta_2<T$ is satisfied, then the LO represents a viable technique. Under this constraint, if δ_5 (which is shorter than δ_2 as already seen in Section II-A.3) is such that the control transmission remains inside the proper time window, the current estimation is feasible and provides a more accurate result.

Remark 13: The physical constraints (concerning the computation time) discussed above provide necessary conditions for implementation. However, taking into account combined packet losses in both communication channels, as well as considering the actuation delay, the IH OFC is not easy to be modeled and formally solved. Trivially, when all the communication is lost, an unstable plant cannot be stabilized remotely. The conditions concerning the theoretical existence of the IH estimators and controllers operating over FSMCs can be based on the *MS* detectability and stabilizability notions (discussed in the following sections) guaranteeing a *MS* stable behavior of estimators and controller with pre-computed gains.

V. Observer stability analysis

This section provides the *MSD* specializations for the LO and the CE, respectively.

A. The operators

We introduce some mathematical preliminaries instrumental for *MSS* analysis (see [11]). For all $\mathbf{S}=[S_m]_{m=1}^I$, $\mathbf{T}=[T_m]_{m=1}^I$, both in $\mathbb{H}^{In_x}$, we specify the inner product as $\langle\mathbf{S};\mathbf{T}\rangle\triangleq\sum_{m=1}^I\mathrm{tr}\,(\mathrm{S}_\mathrm{m}^*\mathrm{T}_\mathrm{m})$. Let us define the operators $\mathcal{E}(\cdot)\triangleq[\mathcal{E}_m(\cdot)]_{m=1}^I$, $\mathcal{D}(\cdot)\triangleq[\mathcal{D}_m(\cdot)]_{m=1}^I$, $\mathcal{T}(\cdot)\triangleq[\mathcal{T}_m(\cdot)]_{m=1}^I$, $\mathcal{L}(\cdot)\triangleq[\mathcal{L}_m(\cdot)]_{m=1}^I$, and $\mathcal{V}(\cdot)\triangleq[\mathcal{V}_m(\cdot)]_{m=1}^I$, all in $\mathbb{B}\left(\mathbb{H}^{In_x}\right)$, for all $\mathbf{S}=[S_m]_{m=1}^I$ in $\mathbb{H}^{In_x}$, $m,n\in\mathbb{S}_\eta$, as follows.

$$\mathcal{E}_m(\mathbf{S})\triangleq\sum\nolimits_{n=1}^{I}q_{mn}S_n,\quad \mathcal{D}_n(\mathbf{S})\triangleq\sum\nolimits_{m=1}^{I}q_{mn}S_m, \quad (17)$$

$$\mathcal{T}_n(\mathbf{S})\triangleq\sum\nolimits_{m=1}^{I}q_{mn}\left\{\hat{\gamma}_m\widehat{\Gamma}_{m1}S_m\widehat{\Gamma}_{m1}^*+(1-\hat{\gamma}_m)\widehat{\Gamma}_{m0}S_m\widehat{\Gamma}_{m0}^*\right\}, \quad (18)$$

$$\mathcal{L}_m(\mathbf{S})\triangleq\hat{\gamma}_m\Gamma_{m1}^*\mathcal{E}_m(\mathbf{S})\Gamma_{m1}+(1-\hat{\gamma}_m)\Gamma_{m0}^*\mathcal{E}_m(\mathbf{S})\Gamma_{m0}, \quad (19)$$

$$\mathcal{V}_n(\mathbf{S})\triangleq\hat{\gamma}_n\Gamma_{n1}\mathcal{D}_n(\mathbf{S})\Gamma_{n1}^*+(1-\hat{\gamma}_n)\Gamma_{n0}\mathcal{D}_n(\mathbf{S})\Gamma_{n0}^*, \quad (20)$$

where the matrices Γ_{n1}, Γ_{n0}, $\widehat{\Gamma}_{n1}$, and $\widehat{\Gamma}_{n0}$ are arbitrary matrices in $\mathbb{F}^{n_x\times n_x}$ that will be specialized later in the paper, while q_{mn} and $\hat{\gamma}_n$ are those defined by (2) and (3), respectively. Define $\mathcal{O}(\cdot,\cdot):\mathbb{H}^{In_x,n_y}\times\mathbb{R}^I\to\mathbb{H}^{In_x}$, with $\mathcal{O}(\cdot,\cdot)\triangleq[\mathcal{O}_m(\cdot,\cdot)]_{m=1}^I$, and $\widehat{\mathcal{O}}(\cdot,\cdot):\mathbb{H}^{In_x,n_y}\times\mathbb{R}^I\to\mathbb{H}^{In_x}$, with $\widehat{\mathcal{O}}(\cdot,\cdot)\triangleq[\widehat{\mathcal{O}}_m(\cdot,\cdot)]_{m=1}^I$,

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

for $\mathbf{M}\triangleq[M_n]_{n=1}^I$ arbitrary matrix in $\mathbb{H}^{In_x,n_y}$, and $\boldsymbol{\alpha}=[\alpha_n]_{n=1}^I$ arbitrary vector in $\mathbb{R}^I$, $n\in\mathbb{S}_\eta$, as

$$\mathcal{O}_n(\mathbf{M},\boldsymbol{\alpha})\triangleq\alpha_n\left(GG^*+\hat{\gamma}_n M_n HH^* M_n^*\right), \quad (21)$$

$$\widehat{\mathcal{O}}_n(\mathbf{M},\boldsymbol{\alpha})\triangleq\sum_{m=1}^{I} q_{mn}\alpha_m(GG^*+\hat{\gamma}_m AM_mHH^*M_m^*A^*). \quad (22)$$

Given $\mathcal{K}_{m\kappa}=\bar{\Gamma}_{m\kappa}\otimes\Gamma_{m\kappa}$, $\kappa=0,1$, define $\mathcal{C},\mathcal{N}\in\mathbb{F}^{In_x^2\times In_x^2}$, as

$$\mathcal{N}\triangleq\bigoplus_{m=1}^{I}(\hat{\gamma}_m\mathcal{K}_{m1})+\bigoplus_{m=1}^{I}((1-\hat{\gamma}_m)\mathcal{K}_{m0}),\ \mathcal{C}\triangleq Q'\otimes\mathbb{I}_{n_x^2}. \quad (23)$$

Remark 14: The matrices $\mathcal{N}$ and $\mathcal{C}$ are designed with the aim of providing a suitable methodology for the test of detectability conditions in Definitions 4 and 5, as will be discussed later. However, even though the aim is the same as in [11], differently from [11], they account for the general FSMC scenario, i.e., they involve the probability $\hat{\gamma}_m$, $m\in\mathbb{S}_\eta$.

Proposition 1: Consider the operators $\mathcal{T}$, $\mathcal{L}$, $\mathcal{V}$ in $\mathbb{B}(\mathbb{H}^{In_x})$, defined in (18), (19), and (20), respectively. Then,

i) $\rho(\mathcal{L})=\rho(\mathcal{V})$,

ii) if $\widehat{\Gamma}_{m0}=\Gamma_{m0}$ and $\widehat{\Gamma}_{m1}=\Gamma_{m1}$ for all $m\in\mathbb{S}_\eta$, then $\rho(\mathcal{L})=\rho(\mathcal{V})=\rho(\mathcal{T})$.

Proof: See Appendix. ∎

Remark 15: Proposition 1 shows the equivalence of operators $\mathcal{V}$, $\mathcal{T}$, $\mathcal{L}$, concerning the spectral radius [11, Ch. 3].

B. Luenberger observer stability analysis

The Luenberger stability analysis is based on the IH solution of filtering CAREs, which are derived as on the asymptotic solution of difference Riccati equations and obtained by defining the first and second moments of the error $\check{e}_k, k\in\mathbb{N}$, as follows,

$$\check{m}_n(k)\triangleq\mathbb{E}\left[\check{e}_k\mathbf{1}_{\{\eta_{k-1}=n\}}\right],\ \check{\mathbf{m}}(k)\triangleq\left[\check{m}_n(k)\right]_{n=1}^I\in\mathbb{F}^{In_x}, \quad (24)$$

$$\check{Y}_n(k)\triangleq\mathbb{E}\left[\check{e}_k\check{e}_k^*\mathbf{1}_{\{\eta_{k-1}=n\}}\right],\ \check{\mathbf{Y}}(k)\triangleq[\check{Y}_n(k)]_{n=1}^I\in\mathbb{H}^{In_x,+}; \quad (25)$$

for $n\in\mathbb{S}_\eta$. Consequently, $\mathbb{E}[\check{e}_k]=\sum_{n=1}^I\check{m}_n(k)$ and $\mathbb{E}[\check{e}_k\check{e}_k^*]=\sum_{n=1}^I\check{Y}_n(k)$. For arbitrary matrices Γ_{n1} and Γ_{n0} in $\mathbb{F}^{n_x\times n_x}$, $n\in\mathbb{S}_\eta$, define $\check{\mathcal{B}}\in\mathbb{F}^{In_x\times In_x}$ as

$$\check{\mathcal{B}}\triangleq\left(\left(\bigoplus_{n=1}^{I}(\hat{\gamma}_n\Gamma_{n1}\right)+\left(\bigoplus_{n=1}^{I}((1-\hat{\gamma}_n)\Gamma_{n0})\right)\right)(Q'\otimes\mathbb{I}_{n_x}). \quad (26)$$

Define also $\check{\mathbf{M}}\triangleq[\check{M}_m]_{m=1}^I$, i.e., the sequence of mode-dependent filtering gains in (10) providing the solution of the LO filtering problem. Hence, we can state the following.

Proposition 2: Consider the error system described by (11). Then, for all $k\in\mathbb{N}$, the following equalities hold:

$$\check{\mathbf{m}}(k+1)=\check{\mathcal{B}}\check{\mathbf{m}}(k), \check{\mathbf{Y}}(k+1)=\mathcal{V}(\check{\mathbf{Y}}(k))+\mathcal{O}(\check{\mathbf{M}},\boldsymbol{\pi}(k)), \quad (27)$$

with $\check{\mathcal{B}}$, $\mathcal{V}$, and $\mathcal{O}$ defined in (26), (20) and (21), for $\Gamma_{n0}=A$ and $\Gamma_{n1}=A+\check{M}_nL$, $n\in\mathbb{S}_\eta$.

Proof: See Appendix. ∎

The following definition provides a specialization of Definition 3 for the LO scenario.

Definition 4 (MSD): The system described by (5) is *MSD* if, for each mode $n\in\mathbb{S}_\eta$, there exists a mode-dependent filtering gain $\check{M}_n\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{V})<1$, $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ defined in (20), for $\Gamma_{n1}=A+\check{M}_nL$ and $\Gamma_{n0}=A$.

From now on, we refer to Definition 4 when using *MSD*.

Remark 16: By applying the results from [11, Sec. 3.4.2], the property provided by Definition 4 is equivalent to the *MSS* of the error system (11).

C. Current estimator stability analysis

Analogous steps for the Luenberger observer stability analysis are reported in the following. Define for $n\in\mathbb{S}_\eta$, $k\in\mathbb{N}$,

$$m_n(k)\triangleq\mathbb{E}\left[e_k\mathbf{1}_{\{\eta_k=n\}}\right],\ \mathbf{m}(k)\triangleq[m_n(k)]_{n=1}^I\in\mathbb{F}^{In_x}, \quad (28)$$

$$Z_n(k)\triangleq\mathbb{E}\left[e_ke_k^*\mathbf{1}_{\{\eta_k=n\}}\right],\ \mathbf{Z}(k)\triangleq\left[Z_n(k)\right]_{n=1}^I\in\mathbb{H}^{In_x,+}. \quad (29)$$

Consequently, $\mathbb{E}[e_k]$ and $\mathbb{E}[e_ke_k^*]$ are given by $\mathbb{E}[e_k]=\sum_{n=1}^I m_n(k)$ and $\mathbb{E}[e_ke_k^*]=\sum_{n=1}^I Z_n(k)$. For $\widehat{\Gamma}_{n1}$ and $\widehat{\Gamma}_{n0}\in\mathbb{F}^{n_x\times n_x}$, $n\in\mathbb{S}_\eta$, define $\widehat{\mathcal{B}}\in\mathbb{F}^{In_x\times In_x}$ as

$$\widehat{\mathcal{B}}\triangleq(Q'\otimes\mathbb{I}_{n_x})\left(\bigoplus_{n=1}^{I}\left(\hat{\gamma}_n\widehat{\Gamma}_{n1}\right)+\bigoplus_{n=1}^{I}\left((1-\hat{\gamma}_n)\widehat{\Gamma}_{n0}\right)\right). \quad (30)$$

Let $\widehat{\mathbf{M}}=[\widehat{M}_m]_{m=1}^I$ be a sequence of mode-dependent filtering gains in (12) providing the solution of the CE filtering problem. The following proposition formalizes the dynamics of the observation error first and second moments.

Proposition 3: Consider the error system described by (15). Then, for all $k\in\mathbb{N}$, the following equalities hold:

$$\mathbf{m}(k+1)=\widehat{\mathcal{B}}\mathbf{m}(k),\ \mathbf{Z}(k+1)=\mathcal{T}(\mathbf{Z}(k))+\widehat{\mathcal{O}}\left(\widehat{\mathbf{M}},\boldsymbol{\pi}(k)\right), \quad (31)$$

with $\widehat{\mathcal{B}}$, $\mathcal{T}$, and $\widehat{\mathcal{O}}$ defined in (30), (18), and (22), respectively, for $\widehat{\Gamma}_{n1}=A+A\widehat{M}_nL$ and $\widehat{\Gamma}_{n0}=A$, $n\in\mathbb{S}_\eta$.

Proof: See Appendix. ∎

The following definition adapts Definition 3 to the CE scenario.

Definition 5 (Strict-MSD): The system described by (5) is *Strict-MSD* if, for each mode $n\in\mathbb{S}_\eta$, there exists a mode-dependent filtering gain $\widehat{M}_n\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{T})<1$, with $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{In_x})$ defined in (18), for $\widehat{\Gamma}_{n1}=A+A\widehat{M}_nL$ and $\widehat{\Gamma}_{n0}=A$.

Proposition 4: Assume that MJLS (5) is *Strict-MSD*. Then, (5) is *MSD* according to Definition 4.

Proof: See Appendix. ∎

Remark 17: By the results from [11, Sec. 3.4.2] applied to the operator $\mathcal{T}$ (with $\mathcal{T}$ as in Definition 5), $\rho(\mathcal{T})<1$ is equivalent to the *MSS* of the error system described by (15).

D. The Luenberger observer filtering CAREs

The optimal mode-dependent filtering gain of LO results from the optimization of the following performance index: $J_L^*=\limsup_{t\to\infty}(1/t)\mathbb{E}[\sum_{k=0}^t(\check{e}_k\check{e}_k^*)\mid\mathcal{F}_l^k]$. Obtaining the optimal performance index in the Luenberger scenario necessitates dealing with Luenberger filtering CAREs, introduced as follows. Define for any $\mathbf{Y}\in\mathbb{H}^{In_x,*}$, $\boldsymbol{\alpha}=[\alpha_n]_{n=1}^I\in\mathbb{R}^I$,

$$\check{\mathcal{A}}_n(\mathbf{Y},\boldsymbol{\alpha})\triangleq A\mathcal{D}_n(\mathbf{Y})A^*+\alpha_nGG^*,\quad \check{\mathcal{B}}_n(\mathbf{Y})\triangleq A\mathcal{D}_n(\mathbf{Y})L^*,$$

$$\check{\mathcal{R}}_n(\mathbf{Y},\boldsymbol{\alpha})\triangleq\alpha_nHH^*+L\mathcal{D}_n(\mathbf{Y})L^*,\quad \check{\mathcal{C}}_n(\mathbf{Y})\triangleq\hat{\gamma}_n^{\frac{1}{2}}\check{\mathcal{B}}_n(\mathbf{Y}),$$

for $n\in\mathbb{S}_\eta$. Consider the set $\mathbb{W}$, defined as follows:

$$\mathbb{W}=\{(\mathbf{Y},\boldsymbol{\alpha})\in\mathbb{H}^{In_x,*}\times\mathbb{R}^I,\ \text{such that}\ \check{\mathcal{R}}_n(\mathbf{Y},\boldsymbol{\alpha})\ \text{is non-singular for any}\ n\in\mathbb{S}_\eta\}.$$

For $(\mathbf{Y},\boldsymbol{\alpha})\in\mathbb{W}$, define the operators $\mathcal{M}(\cdot,\cdot):\mathbb{W}\to\mathbb{H}^{In_x,n_y}$ and $\mathcal{Y}(\cdot,\cdot):\mathbb{W}\to\mathbb{H}^{In_x}$ as $\mathcal{M}(\mathbf{Y},\boldsymbol{\alpha})=\left[\mathcal{M}_n(\mathbf{Y},\boldsymbol{\alpha})\right]_{n=1}^{I}$ and $\mathcal{Y}(\mathbf{Y},\boldsymbol{\alpha})=\left[\mathcal{Y}_n(\mathbf{Y},\boldsymbol{\alpha})\right]_{n=1}^{I}$, with

$$\mathcal{M}_n(\mathbf{Y},\boldsymbol{\alpha})\triangleq-\check{\mathcal{B}}_n(\mathbf{Y})\,\check{\mathcal{R}}_n^{-1}(\mathbf{Y},\boldsymbol{\alpha}), \tag{32}$$

$$\mathcal{Y}_n(\mathbf{Y},\boldsymbol{\alpha})\triangleq\check{\mathcal{A}}_n(\mathbf{Y},\boldsymbol{\alpha})-\check{\mathcal{C}}_n(\mathbf{Y})\check{\mathcal{R}}_n^{-1}(\mathbf{Y},\boldsymbol{\alpha})\,\check{\mathcal{C}}_n^*(\mathbf{Y}), \tag{33}$$

for any $n\in\mathbb{S}_\eta$ (see [11, Sec. A.1]). For notational convenience, let us set $\mathcal{M}(\mathbf{Y})=\mathcal{M}(\mathbf{Y},\boldsymbol{\pi}^\infty)$, $\mathcal{Y}(\mathbf{Y})=\mathcal{Y}(\mathbf{Y},\boldsymbol{\pi}^\infty)$, and, for $n\in\mathbb{S}_\eta$, $\check{\mathcal{R}}_n(\mathbf{Y})=\check{\mathcal{R}}_n(\mathbf{Y},\boldsymbol{\pi}^\infty)$, $\check{\mathcal{A}}_n(\mathbf{Y})=\check{\mathcal{A}}_n(\mathbf{Y},\boldsymbol{\pi}^\infty)$. The LO filtering CAREs are the set of equations given by

$$Y_n=\mathcal{Y}_n(\mathbf{Y}),\ n\in\mathbb{S}_\eta. \tag{34}$$

The optimal IH mode-dependent filtering gain is obtained from the solution of the following optimization problem:

$$\max\ \mathrm{tr}\left(\sum_{n=1}^{I}Y_n\right) \tag{35a}$$

subject to

$$\begin{bmatrix}-Y_n+\check{\mathcal{A}}_n(\mathbf{Y}) & \check{\mathcal{C}}_n(\mathbf{Y})\\ \check{\mathcal{C}}_n^*(\mathbf{Y}) & \check{\mathcal{R}}_n(\mathbf{Y})\end{bmatrix}\succeq 0, \tag{35b}$$

$$\check{\mathcal{R}}_n(\mathbf{Y})\succ 0,\ \mathbf{Y}\in\mathbb{H}^{In_x,*},\ n\in\mathbb{S}_\eta. \tag{35c}$$

Define the sets $\mathbb{L}$ and $\mathbb{M}$ as follows,

$$\mathbb{L}\triangleq\{\mathbf{Y}\in\mathbb{H}^{In_x,*};\ \check{\mathcal{R}}_n(\mathbf{Y})\ \text{non-singular}\ \forall n\in\mathbb{S}_\eta\},$$
$$\mathbb{M}\triangleq\{\mathbf{Y}\in\mathbb{L};\ \check{\mathcal{R}}(\mathbf{Y})\succ 0\ \text{and}\ -\mathbf{Y}+\mathcal{Y}(\mathbf{Y})\succeq 0\}.$$

Then, the *MS* stabilizing filtering gain is given by

$$\check{M}_n=\mathcal{M}_n(\mathbf{Y}),\ n\in\mathbb{S}_\eta, \tag{36}$$

where $\mathbf{Y}\in\mathbb{L}$ is the *MS* stabilizing solution of (34) [11, Sec. A.1].

Definition 6 (MS stabilizing solution of (34)*):* $\mathbf{Y}\in\mathbb{L}$ is the *MS* stabilizing solution of (34) if it satisfies (34) and $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ defined in (20), $\Gamma_{n1}=A+\mathcal{M}_n(\mathbf{Y})L$ and $\Gamma_{n0}=A$, $n\in\mathbb{S}_\eta$; i.e., $\mathcal{M}_n(\mathbf{Y})$ stabilizes the error system (11) in the *MS* sense.

The maximal solution of (34) and the solution of (35) coincide, as stated in the following theorem.

Theorem 1: Assume that (5) is *MSD*. Then, the following statements are equivalent:

i) there exists $\mathbf{Y}^+\in\mathbb{M}$ satisfying (34), such that $\mathbf{Y}^+\succeq\mathbf{Y}$, for all $\mathbf{Y}\in\mathbb{M}$,

ii) there exists a solution $\widehat{\mathbf{Y}}$ for the convex programming problem described in (35).

Moreover, the two solutions coincide, i.e., $\widehat{\mathbf{Y}}=\mathbf{Y}^+$.

Proof: See Appendix. ■

The maximal solution and the *MS* stabilizing solution of (34) are connected, as stated in the following theorem.

Theorem 2: There exists at most one *MS* stabilizing solution of (34), which coincides with the maximal solution in $\mathbb{M}$, that is, the solution of the convex programming problem described in (35).

Proof: See Appendix. ■

The *MS* stabilizing filtering gain (36) is computed exploiting the maximal solution of (34), i.e., the solution of (35), as stated in Theorem 2. Consequently, the optimal performance index achieved by the LO is $J_L^*=\sum_{m=1}^{I}\mathrm{tr}(Y_m)$, with $\mathbf{Y}=[Y_m]_{m=1}^{I}$ being the maximal solution of (34). The necessary condition for the existence of the *MS* stabilizing solution of the filtering CAREs is the *MSD* of system (5).

E. The current estimator filtering CAREs

The optimal mode-dependent filtering gain of the CE results from the optimization of the following performance index, $J_C^*=\limsup_{t\to\infty}(1/t)\mathbb{E}[\sum_{k=0}^{t}(e_ke_k^*)|\mathcal{F}_c^k]$.

Remark 18: J_C^* (computed exploiting the prediction error) can be compared to the Luenberger performance index J_L^* (computed exploiting the estimation error) because the estimation error for the LO and the prediction error for the CE are equivalent at the steady-state, see Remark 11.

For $\mathbf{Z}=[Z_m]_{m=1}^{I}\in\mathbb{H}^{In_x,*}$ and $\boldsymbol{\alpha}=[\alpha_n]_{n=1}^{I}\in\mathbb{R}^I$, define $\widehat{\mathcal{A}}_n(\mathbf{Z},\boldsymbol{\alpha})$, $\widehat{\mathcal{R}}_n(\mathbf{Z},\boldsymbol{\alpha})$, and $\widehat{\mathcal{C}}_n(\mathbf{Z})$, as $\widehat{\mathcal{A}}_n(\mathbf{Z},\boldsymbol{\alpha})\triangleq AZ_nA^*+\alpha_nGG^*$, $\widehat{\mathcal{R}}_n(\mathbf{Z},\boldsymbol{\alpha})\triangleq LZ_nL^*+\alpha_nHH^*$, and $\widehat{\mathcal{C}}_n(\mathbf{Z})\triangleq AZ_nL^*$, respectively, for $n\in\mathbb{S}_\eta$. Consider the set $\mathbb{W}_c$,

$$\mathbb{W}_c=\{(\mathbf{Z},\boldsymbol{\alpha})\in\mathbb{H}^{In_x,*}\times\mathbb{R}^I,\ \text{such that}\ \widehat{\mathcal{R}}_n(\mathbf{Z},\boldsymbol{\alpha})\ \text{is non-singular for any}\ n\in\mathbb{S}_\eta\}.$$

For $(\mathbf{Z},\boldsymbol{\alpha})\in\mathbb{W}_c$, define operators $\widehat{\mathcal{M}}(\cdot,\cdot):\mathbb{W}_c\to\mathbb{H}^{In_x,n_y}$ and $\mathcal{Z}(\cdot,\cdot):\mathbb{W}_c\to\mathbb{H}^{In_x}$ as $\widehat{\mathcal{M}}(\mathbf{Z},\boldsymbol{\alpha})=[\widehat{\mathcal{M}}_n(\mathbf{Z},\boldsymbol{\alpha})]_{n=1}^{I}$ and $\mathcal{Z}(\mathbf{Z},\boldsymbol{\alpha})=[\mathcal{Z}_n(\mathbf{Z},\boldsymbol{\alpha})]_{n=1}^{I}$ [11, Sec. A.1], with

$$\mathcal{Z}_n(\mathbf{Z},\boldsymbol{\alpha})\triangleq\sum_{m=1}^{I}q_{mn}\{\widehat{\mathcal{A}}_m(\mathbf{Z},\boldsymbol{\alpha})-\hat{\gamma}_m\widehat{\mathcal{C}}_m(\mathbf{Z})\widehat{\mathcal{R}}_m^{-1}(\mathbf{Z},\boldsymbol{\alpha})\widehat{\mathcal{C}}_m^*(\mathbf{Z})\},$$
$$\widehat{\mathcal{M}}_n(\mathbf{Z},\boldsymbol{\alpha})\triangleq-Z_nL^*\widehat{\mathcal{R}}_n^{-1}(\mathbf{Z},\boldsymbol{\alpha}).$$

For notational convenience, let us set $\widehat{\mathcal{M}}(\mathbf{Z})=\widehat{\mathcal{M}}(\mathbf{Z},\boldsymbol{\pi}^\infty)$ and $\mathcal{Z}(\mathbf{Z})=\mathcal{Z}(\mathbf{Z},\boldsymbol{\pi}^\infty)$, that are CE filtering CAREs.

The following lemma states the equivalence of the filtering CAREs solutions and the filtering gains, for the LO and CE.

Lemma 1: The following statements are equivalent:

i) For any $\mathbf{Y}(0)\in\mathbb{H}^{In_x,+}$, $\mathbf{Y}(k)\in\mathbb{H}^{In_x,+}$, $k\in\mathbb{N}$, satisfying $\mathbf{Y}(k+1)=\mathcal{Y}(\mathbf{Y}(k),\boldsymbol{\pi}(k))$, with $\mathcal{Y}$ defined in (33), converges to $\mathbf{Y}\in\mathbb{H}^{In_x,+}$ satisfying $\mathbf{Y}=\mathcal{Y}(\mathbf{Y})$.

ii) For any $\mathbf{Z}(0)\in\mathbb{H}^{In_x,+}$, $\mathbf{Z}(k)\in\mathbb{H}^{In_x,+}$, $k\in\mathbb{N}$, satisfying $\mathbf{Z}(k+1)=\mathcal{Z}(\mathbf{Z}(k),\boldsymbol{\pi}(k))$, converges to $\mathbf{Z}\in\mathbb{H}^{In_x,+}$ satisfying $\mathbf{Z}=\mathcal{Z}(\mathbf{Z})$.

Moreover, the mode-dependent filtering gain that stabilizes the error system (15) in the *MS* sense is $\widehat{M}_n=\widehat{\mathcal{M}}_n(\mathbf{Z})$, and the optimal performance index achieved by the CE is $J_C^*=\sum_{n=1}^{I}\mathrm{tr}(Z_n)$, with $\mathbf{Z}=[Z_n]_{n=1}^{I}\in\mathbb{H}^{In_x}$, Z_n given by $Z_n=\mathcal{D}_n(\mathbf{Y})$, $n\in\mathbb{S}_\eta$, and $\mathbf{Y}$ maximal solution of (34).

Proof: See Appendix. ■

Remark 19: The LO and the CE are equivalent from the steady-state point of view, as stated in Lemma 1. However, their difference in performance (indicated by indexes J_L and J_C) and physical constraints (see Remark 13) allow for choosing the most suitable estimator for a specific scenario, as shown in Sections II-A.3 and IV-C.

Remark 20: If the matrix A is non-singular, then, from Lemma 1, we may compute the LO filtering gain as $\check{M}_n=A^{-1}\widehat{M}_n$.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

VI. The separation principle

In the following we state the separation principle for the LO and CE scenarios, respectively.

A. The Luenberger observer separation principle

Consider the optimal matrices in (8), which we can express as follows:

$$\check{A}(\nu_k,\theta_{k-1},\gamma_k,\eta_k) = A + \nu_k B F_{\theta_{k-1}} + \gamma_k \check{M}_{\eta_k} L,$$
$$\check{B}(\eta_k) = -\check{M}_{\eta_k}, \quad \check{F}(\theta_{k-1}) = F_{\theta_{k-1}}.$$

Then, the optimal output-feedback controller (8) coincides with (10), and the closed-loop system dynamics are

$$x_{k+1}=(A+\nu_k BF_{\theta_{k-1}})x_k-\nu_k BF_{\theta_{k-1}}\check{e}_k+Gw_k. \quad (37)$$

By recalling the error dynamics described in (11), we write the closed-loop system as follows:

$$\check{\mathcal{G}}_{cl}:\ \mathcal{E}_{k+1} = \mathbf{\Gamma}(\nu_k,\theta_{k-1},\gamma_k,\eta_k)\,\mathcal{E}_k + \mathbf{\Sigma}(\gamma_k,\eta_k)w_k, \quad (38)$$

$$\mathcal{E}_k \triangleq \begin{bmatrix} x_k \\ \check{e}_k \end{bmatrix}, \qquad \mathbf{\Sigma}(\gamma_k,\eta_k) \triangleq \begin{bmatrix} G \\ G+\gamma_k\check{M}_{\eta_k}H \end{bmatrix},$$
$$\mathbf{\Gamma}(\nu_k,\theta_{k-1},\gamma_k,\eta_k) \triangleq \begin{bmatrix} (A+\nu_k BF_{\theta_{k-1}}) & -\nu_k BF_{\theta_{k-1}} \\ \mathbb{O}_{n_x} & (A+\gamma_k\check{M}_{\eta_k}L) \end{bmatrix}.$$

Theorem 3: Given a MJLS described by (5) and the LO (10), the following statements are equivalent:

$i)$ the dynamics (37) can be made *MSS*;
$ii)$ the MJLS described by (5) is both
- ii-$a)$ *MSD*,
- ii-$b)$ *MS* stabilizable with one time-step delayed AL mode observation.

Proof: See Appendix. ■

B. The current estimator separation principle

Consider the optimal matrices in (9), which we assert as follows:

$$\widehat{A}(\gamma_k,\eta_k)\triangleq A+\gamma_k A\widehat{M}_{\eta_k}L,\ \ \widehat{B}(\eta_k)\triangleq -A\widehat{M}_{\eta_k},$$
$$\widehat{F}(\theta_{k-1})\triangleq F_{\theta_{k-1}},\ \ \widehat{C}(\nu_k,\theta_{k-1})\triangleq \nu_k BF_{\theta_{k-1}},\ \ \widehat{D}(\eta_k)\triangleq -\widehat{M}_{\eta_k}.$$

Then, (12)-(14) coincide with (9), and the dynamics of the closed-loop system are the following:

$$x_{k+1}=(A+\nu_k BF_{\theta_{k-1}})x_k+(G-\gamma_k\nu_k BF_{\theta_{k-1}}\widehat{M}_{\eta_k}H)w_k$$
$$-(\nu_k BF_{\theta_{k-1}}+\gamma_k\nu_k BF_{\theta_{k-1}}\widehat{M}_{\eta_k}L)e_k. \quad (39)$$

By recalling the error dynamics described in (15), we write the closed-loop system in a compact form as follows:

$$\widehat{\mathcal{G}}_{cl}:\ \mathcal{X}_{k+1} = \mathbf{\Psi}(\nu_k,\theta_{k-1},\gamma_k,\eta_k)\,\mathcal{X}_k + \mathbf{\Omega}(\nu_k,\theta_{k-1},\gamma_k,\eta_k)\,w_k,$$

with $\mathbf{\Psi}(\nu_k,\theta_{k-1},\gamma_k,\eta_k)\triangleq$

$$\begin{bmatrix} (A+\nu_k BF_{\theta_{k-1}}) & -(\nu_k BF_{\theta_{k-1}}+\gamma_k\nu_k BF_{\theta_{k-1}}\widehat{M}_{\eta_k}L) \\ \mathbb{O}_{n_x} & (A+\gamma_k A\widehat{M}_{\eta_k}L) \end{bmatrix},$$

$$\mathcal{X}_k\triangleq\begin{bmatrix} x_k \\ e_k \end{bmatrix}, \quad \mathbf{\Omega}(\nu_k,\theta_{k-1},\gamma_k,\eta_k)\triangleq\begin{bmatrix} G-\gamma_k\nu_k BF_{\theta_{k-1}}\widehat{M}_{\eta_k}H \\ G+\gamma_k A\widehat{M}_{\eta_k}H \end{bmatrix}.$$

TABLE I
Detectability and stabilizability analysis summary

	C.A	C.B	C.C	C.D
MSD	✗	✓	✓	✓
Strict-MSD	✗	✓	✓	✓
Strong-MSD	✗	✗	✓	✓
Strong-Strict-MSD	✗	✗	✓	✓
MS stabilizability	✗	✗	✓	✓
Strong-*MS* stabilizability	✗	✗	✗	✓

Remark 21: The matrices $\mathbf{\Psi}$ and $\mathbf{\Gamma}$ are upper triangular block diagonal matrices as in [11], i.e., the error dynamics (driven by $\{\eta_k\}$) do not depend on the state dynamics (induced by $\{\theta_k\}$). Differently from [11], the closed-loop dynamical matrices $\mathbf{\Gamma}$ and $\mathbf{\Psi}$ contain the Markov jumps not only of the Markov chain $\{\eta_k\}$ (SL) but of the Markov chain $\{\theta_k\}$ (AL) too (see the FSMC model in Sections II-A.1 and II-A.2). Moreover, we consider the mode observation delay affecting the Markov chain $\{\theta_k\}_{k\in\mathbb{N}}$.

Theorem 4: Given a MJLS described by (5) and CE (12), the following statements are equivalent:

$i)$ the dynamics (39) can be made *MSS*,
$ii)$ the MJLS described by (5) is both
- ii-$a)$ *Strict-MSD*,
- ii-$b)$ *MS* stabilizable with one time-step delayed AL mode observation.

Proof: See Appendix. ■

VII. Mode-independent output-feedback

Under the conditions presented in this section, the designer can use mode-independent control and filtering gains. The advantage of mode-independence concerns the reduced computational burden, especially when the number of modes increases. The strong *MS* stabilizability (defined in the following) guarantees the existence of a mode-independent control gain, which is *MS* stabilizing. On the other hand, the following definitions of *Strong-MSD* and *Strong-Strict-MSD* provide the basis for deriving sufficient conditions guaranteeing the existence of a mode-independent filtering gain, which makes the estimation error system *MSS*.

Definition 7 (Strong-MS stabilizability): The system (5) is Strong-*MS* stabilizable with one time-step delayed AL mode observation if, for any initial condition (x_0,θ_0), there exists a mode-independent control gain $F^b\in\mathbb{F}^{n_u\times n_x}$ such that $u_k=F^b x_k$ is the *MS* stabilizing state-feedback for (5).
The following *Strong-MSD* and *Strong-Strict-MSD* notions instead concern the SL.

Definition 8 (Strong-MSD): The system (5) is *Strong-MSD* if there exists a mode-independent filtering gain $\check{M}^b\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{I n_x})$ defined in (20), for $\Gamma_{n1}=A+\check{M}^b L$, $\Gamma_{n0}=A$, and $n\in\mathbb{S}_\eta$.

Definition 9 (Strong-Strict-MSD): The system (5) is *Strong-Strict-MSD* if there exists a mode-independent filtering gain $\widehat{M}^b\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{T})<1$, with $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{I n_x})$ defined in (18), for $\widehat{\Gamma}_{n1}=A+A\widehat{M}^b L$, $\widehat{\Gamma}_{n0}=A$, and $n\in\mathbb{S}_\eta$.

Proposition 5: Consider the MJLS (5). The following implications hold.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

Fig. 4. Estimation error on cart position obtained by Monte Carlo simulations are reported in yellow, the mean error trajectory in red, the maximum error trajectory in blue, and the minimum error trajectory in green. The top right of each panel reports a zoom in for each plot.

$i)$ *Strong-MSD* implies *MSD*.

$ii)$ *Strong-Strict-MSD* implies *Strict-MSD* and *Strong-MSD*.

Proof: See Appendix. ■

Remark 22: *Strong-Strict-MSD* implies all the detectability notions concerning the FSMC model. Thus, it is the strongest notion, while *MSD* is the weakest one.

We introduce the mode-independent output-feedback recalling the filtering and control modified algebraic Riccati equations (MARE) reported in the following [4], [26]. To this end, define

$$\mathring{\mathcal{A}}^b(Y^b) \triangleq AY^bA^* + GG^*, \quad \mathring{\mathcal{C}}^b(Y^b) \triangleq AY^bL^*,$$
$$\mathring{\mathcal{R}}^b(Y^b) \triangleq LY^bL^* + HH^*, \quad \mathring{\mathcal{M}}^b(Y^b) = -Y^bL^*\mathring{\mathcal{R}}^b(Y^b)^{-1},$$
$$\mathcal{A}^b(X^b) \triangleq A^*X^bA + C^*C, \quad \mathcal{C}^b(X^b) \triangleq B^*X^bA,$$
$$\mathcal{R}^b(X^b) \triangleq B^*X^bB + D^*D, \quad \mathcal{F}^b(X^b) = -\mathcal{R}^b(X^b)^{-1}\mathcal{C}^b(X^b).$$

for Y^b, $X^b \in \mathbb{F}_*^{n_x \times n_x}$. Consider the sets

$$\mathring{\mathbb{L}}^b \triangleq \{Y^b \in \mathbb{F}_*^{n_x \times n_x} \text{ such that } \mathring{\mathcal{R}}^b(Y^b) \text{ is non-singular}\},$$
$$\mathbb{L}^b \triangleq \{X^b \in \mathbb{F}_*^{n_x \times n_x} \text{ such that } \mathcal{R}^b(X^b) \text{ is non-singular}\}.$$

For $Y^b_\infty \in \mathring{\mathbb{L}}^b$, $X^b_\infty \in \mathbb{L}^b$, the filtering and control MARE are

$$Y^b_\infty = \mathring{\mathcal{A}}^b(Y^b_\infty) - \mathring{\gamma}\mathring{\mathcal{C}}^b(Y^b_\infty)\mathring{\mathcal{R}}^b(Y^b_\infty)^{-1}\mathring{\mathcal{C}}^{b*}(Y^b_\infty), \quad (40)$$

$$X^b_\infty = \mathcal{A}^b(X^b_\infty) - \mathring{\nu}\mathcal{C}^{b*}(X^b_\infty)\mathcal{R}^b(X^b_\infty)^{-1}\mathcal{C}^b(X^b_\infty). \quad (41)$$

Under the strong *MS* stabilizability condition, the mode-independent *MS* stabilizing control gain exists, and it is given by $F^b = \mathcal{F}^b(X^b_\infty)$, with $X^b_\infty \in \mathbb{L}^b$ satisfying (41) [13]. Moreover, the critical arrival probability on the AL is defined as $\nu_c \triangleq \inf_\nu\{0 \le \nu \le 1 \text{ such that } X^b_\infty \succeq 0 \text{ satisfies (41)}\}$ [4, Lemma 5.4 (a)], and the critical observation arrival probability on the SL is denoted by γ_c [4, Th. 5.5]. By [4, Lemma 5.4, Th. 5.5], ν_c and γ_c satisfy $p_{\min} \le \nu_c \le p_{\max}$ and $p_{\min} \le \gamma_c \le \gamma_{\max} \le p_{\max}$, where $p_{\min} \triangleq 1 - 1/(\max_h |\lambda_h^u(A)|^2)$, $p_{\max} \triangleq 1 - 1/\prod_h |\lambda_h^u(A)|^2$, with $\lambda_h^u(A)$ being the h-th unstable eigenvalue of A, and $\gamma_{\max} \triangleq \inf_\gamma\{0 \le \gamma \le 1 \text{ such that } Y^b_\infty \succeq 0 \text{ satisfies (40)}\}$.

Remark 23: *Strong-MSD* condition guarantees the existence of the mode-independent filtering gain, that can be computed as $\check{M}^b = A\mathring{\mathcal{M}}^b(Y^b_\infty)$, with $Y^b_\infty \in \mathring{\mathbb{L}}^b$ satisfying (40). Moreover, if *Strong-Strict-MSD* is satisfied, the existence of the CE mode-independent filtering gain is guaranteed. In this case, the filtering gain can be computed as follows: $\widehat{M}^b = \mathring{\mathcal{M}}^b(Y^b_\infty)$, with $Y^b_\infty \in \mathring{\mathbb{L}}^b$.

The next theorem links the optimal mode-dependent filtering CARE solution and mode-independent solutions of the filtering MARE. Specifically, the solutions of the filtering problem are equivalent under particular conditions. The same holds for the control problem [13, Th. 3].

Theorem 5: Assume that $\mathring{\nu} = \sum_{i=1}^N \varpi_i \hat{\nu}_i$, $\mathring{\gamma} = \sum_{m=1}^I \pi_m^\infty \hat{\gamma}_m$. Then, the following statements hold.

$i)$ The solution of the filtering MARE provides the mode-independent solution of the filtering CAREs.

$ii)$ The solution of the control MARE provides the mode-independent solution of the control CAREs.

Proof: See Appendix. ■

LMIs guaranteeing *MS* detectability conditions are presented as follows:

$$A^*Z_mA + \hat{\gamma}_mA^*W_{m2}L + \hat{\gamma}_mL^*W_{m2}^*A + \hat{\gamma}_mL^*W_{m3}L - W_{m1} \prec 0; \quad (42a)$$

$$\begin{bmatrix} Z_m & W_{m2} \\ W_{m2}^* & W_{m3} \end{bmatrix} \succeq 0; \quad (42b)$$

$$Z_m \succeq \mathcal{E}_m(\mathbf{W}_1), \quad W_{m1} \succ 0, \quad Z_m \succ 0, \; m \in \mathbb{S}_\eta, \quad (42c)$$

$\mathbf{W}_1 = [W_{m1}]_{m=1}^I$, $\mathbf{Z} = [Z_m]_{m=1}^I$ in $\mathbb{H}^{I n_x,+}$, $\mathbf{W}_2 = [W_{m2}]_{m=1}^I$ in $\mathbb{H}^{I n_x, n_y}$, and $\mathbf{W}_3 = [W_{m3}]_{m=1}^I$ in $\mathbb{H}^{I n_y,+}$.

Proposition 6: Consider the MJLS described by (5) and the following statements.

$i)$ MJLS (5) is *MSD*.

$ii)$ MJLS (5) is *Strict-MSD*.

$iii)$ there exist $\mathbf{W}_1 = [W_{m1}]_{m=1}^I$, $\mathbf{Z} = [Z_m]_{m=1}^I \in \mathbb{H}^{I n_x,+}$, $\mathbf{W}_2 = [W_{m2}]_{m=1}^I \in \mathbb{H}^{I n_x, n_y}$, $\mathbf{W}_3 = [W_{m3}]_{m=1}^I \in \mathbb{H}^{I n_y,+}$, satisfying conditions (42).

Then, $(i) \iff (iii)$, $(ii) \implies (iii)$. Furthermore, if A is non-singular, we have $(ii) \iff (iii)$.

Proof: See Appendix. ■

Consider the following set of LMIs:

$$A^*ZA + \hat{\gamma}_mA^*W_2L + \hat{\gamma}_mL^*W_2^*A + \hat{\gamma}_mL^*W_3L - W_{m1} \prec 0; \quad (43a)$$

$$\begin{bmatrix} Z & W_2 \\ W_2^* & W_3 \end{bmatrix} \succeq 0; \quad (43b)$$

$$Z \succeq \mathcal{E}_m(\mathbf{W}_1), \quad W_{m1} \succ 0, \quad Z \succ 0, \; m \in \mathbb{S}_\eta, \quad (43c)$$

$\mathbf{W}_1 = [W_{m1}]_{m=1}^I$ in $\mathbb{H}^{I n_x,+}$, Z in $\mathbb{F}_+^{n_x \times n_x}$, W_2 in $\mathbb{F}^{n_x \times n_y}$, W_3 in $\mathbb{F}_+^{n_y \times n_y}$.

Proposition 7: Consider the MJLS described by (5) and the following statements.

$i)$ There exist $\mathbf{W}_1 = [W_{m1}]_{m=1}^I$ in $\mathbb{H}^{I n_x,+}$, $Z \in \mathbb{F}_+^{n_x \times n_x}$, $W_2 \in \mathbb{F}^{n_x \times n_y}$, $W_3 \in \mathbb{F}_+^{n_y \times n_y}$, satisfying conditions (43).

$ii)$ MJLS (5) is *Strong-MSD*.

$iii)$ MJLS (5) is *Strong-Strict-MSD*.

Then, $(i) \implies (ii)$. Moreover, if A is non-singular, $(i) \implies (iii)$.

Proof: See Appendix. ■

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

VIII. Numerical case study

This section presents the wireless OFC of an inverted pendulum on a cart [27], controlled remotely over TCP-like lossy sensing and actuation links. The considered cart and pendulum masses are 0.5 and $0.2\,\mathrm{kg}$, inertia about the pendulum mass center is $0.006\ \mathrm{kg{\cdot}m^2}$, distance from the pivot to the pendulum mass center is $0.3\,\mathrm{m}$, coefficient of friction for the cart is 0.1. The system state is defined by $x=\left[\delta\mathrm{x},\delta\dot{\mathrm{x}},\delta\phi,\delta\dot{\phi}\right]'$, with $\delta\mathrm{x}(t)=\mathrm{x}(t)-\mathrm{x}^\star$, $\delta\phi(t)=\phi(t)-\phi^\star$, where x is the cart position, ϕ is the pendulum angle from vertical, $\mathrm{x}^\star$ and $\phi^\star$ are the equilibrium point position and angle. The designed control law aims to stabilize the pendulum in the upright position corresponding to unstable equilibrium point $\mathrm{x}^\star=0\,\mathrm{m}$, $\phi^\star=0\,\mathrm{rad}$. The optimal Markov jump output-feedback controllers (8) and (9) have been applied to the discrete-time linear model derived from the continuous-time nonlinear model by linearization. The state-space model of the system is linearized around the unstable equilibrium point and discretized with sampling period $\mathrm{T_s}=0.01\,\mathrm{s}$. The obtained system matrices can be found in [19]. The process noise is characterized by the covariance matrix $\mathbb{E}[w_k w_k^*]=\Sigma_w$, with $\Sigma_w=2\cdot10^{-6}\mathbb{I}_4$. The state matrix A is unstable since it has an eigenvalue 1.057, but it is easy to verify that $D^*D\succ 0$, the pair (A,B) is controllable, while (A,L) is observable, so the closed-loop system is asymptotically stable if $\nu_k{=}1$ and $\gamma_k{=}1\ \forall k$. Moreover, the necessary conditions for the existence of the *MS* stabilizing solution for the control and filtering CAREs are satisfied. FSMC models with TPMs in $\mathbb{R}^{4\times4}$ describe the double-sided packet loss. These channels are obtained by following the systematic procedure in [22] that accounts for path loss, shadow fading, transmission power control, and interference. The partitioning of the SINR range is based on the values of PEP so that each SINR threshold corresponds to a specific PEP value.

Detectability analysis. The proposed methodology is applied to the study of the *MSD* conditions. Simulation results highlight the existence of a limit case for detectability conditions. When considering the distance between the transmitter-receiver couple of interest $d_0=17.348\,\mathrm{m}$ and distance between the interfering transmitter and receiver of interest $d_{i,1}=9.548\,\mathrm{m}$, the resulting SL TPM is given by

$$Q_1=\begin{bmatrix}0.8855395 & 0.0184352 & 0.0603969 & 0.0356284\\ 0.8825920 & 0.0187857 & 0.0617956 & 0.0368267\\ 0.8820434 & 0.0188504 & 0.0620549 & 0.0370513\\ 0.8806549 & 0.0190134 & 0.0627101 & 0.0376216\end{bmatrix}.$$

The probabilities of receiving the packet in each mode of the SL are denoted by $\hat{\gamma}_1=[0.005,0.5000509,0.9237605,1]$. Conditions (42) are satisfied. From Proposition 6, the system is *MSD* and *Strict-MSD*. As far as strong conditions (43) go, they are not satisfied. From the spectral radius analysis, $\rho\left(\mathcal{V}\right)=\rho\left(\mathcal{T}\right)=0.999999983$ with Markovian filtering, and $\rho\left(\mathcal{V}\right)=\rho\left(\mathcal{T}\right)=1.000000074$ with the Bernoullian filtering. In this case, the condition $\hat{\gamma}>\gamma_{\max}$ from [4, Th. 5.6] is satisfied. However, the system is unstable with the Bernoullian filtering because the system is neither *Strong-MSD* nor *Strong-Strict-MSD*. This limit case reveals that the Bernoullian OFC may fail in making the closed-loop system *MSS* when strong detectability conditions are not satisfied, while the Markovian OFC achieves this aim over the FSMCs.

Stabilizability analysis. The *MS* stabilizability analysis is presented through a limit case: consider $d_0=17.348\,\mathrm{m}$ and $d_{i,2}=10\,\mathrm{m}$. Then, the AL TPM is

$$P_2=\begin{bmatrix}0.8647302 & 0.0208232 & 0.0701174 & 0.0443292\\ 0.8615749 & 0.0211698 & 0.0715631 & 0.0456922\\ 0.8609554 & 0.0212373 & 0.0718457 & 0.0459616\\ 0.8593737 & 0.0214086 & 0.0725659 & 0.0466518\end{bmatrix}.$$

The probabilities of receiving the packet in each mode of the AL are given by $\hat{\nu}_2=[0.006,0.5003405,0.9248986,1]$. Thus, $\rho(\widehat{\mathcal{L}})=1.000388084$ (with $\widehat{\mathcal{L}}$ defined in (44)) using the Bernoullian control gain, while the spectral radius $\rho(\widehat{\mathcal{L}})=0.996248733$ with the Markovian mode-dependent control gain. This case highlights that even though the condition $\hat{\nu}>\nu_c$ from [4, Th. 5.6] is satisfied, the system is unstable with the Bernoullian controller because the system is not Strong-*MS* stabilizable (recall Definition 7), see also Remark 24. The Bernoullian control law is not able to make the closed-loop system *MSS*, while the Markovian control achieves this aim. Varying distances d_i between the interfering transmitter and receiver of interest positioned at $d_0=17.348\,\mathrm{m}$ from its transmitter, we distinguish four cases: C.A ($d_i\leq9.547\,\mathrm{m}$), C.B ($d_i=9.548\,\mathrm{m}$), C.C (d_i going from $9.549\,\mathrm{m}$ to $12.100\,\mathrm{m}$), and C.D ($d_i\geq12.101\,\mathrm{m}$). Table I provides insights on the detectability and stabilizability for each of these cases: the check mark indicates that the notion holds, while the cross mark reveals that its required conditions are not satisfied.

Remark 24: The results presented in this paper are more general with respect to the ones by Schenato et al. [4]. As also pointed out in the detectability and stabilizability analysis, even though in this example the conditions by Schenato et al. are satisfied, the system is not *MSS* with the Bernoullian mode-independent controller. This is because *Strong MS stabilizability* and *Strong-MSD* are not satisfied.

Performance analysis and comparison. Consider distances $d_0=17.348\,\mathrm{m}$, $d_{i,3}=14\,\mathrm{m}$ (corresponding to the case C.D in Table I) and covariance matrix Σ_w described before. The performance indexes obtained by the Markovian LO and CE are $J_L^*=0.0001109$ and $J_C^*=0.0000746$, respectively. The performance index obtained by the Bernoullian observer is $J_B^*=209.8934328$. The reported performance indexes highlight the fact that the presented mode-independent estimation techniques are easier to implement, but their average cost is larger than the one obtained by the Markovian filtering. The spectral radius of $\mathcal{T}$ and $\mathcal{V}$ are the same for both mode-dependent Markovian filters because these estimators are equivalent at the steady-state, see Remark 11. However, the advantage of the CE compared to the LO is that it involves the most recent measurement in the estimation, yielding a smaller performance index. Fig. 5 provides the closed-loop mean square state trajectories obtained with 1000 independent trajectories. As the reader may notice, the CE leads to closed-loop mean square state trajectory that remain far below the closed-loop mean square state trajectory provided by LO. Consider the scenario with distances $d_0=17.348\,\mathrm{m}$, $d_{i,3}=14\,\mathrm{m}$, where $\Sigma_w=qq'$, with $q=[0.003,1,-0.005,-2.150]'$ [4]. This case is reported in Fig. 4 to emphasize the performance differences

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and
content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3328268

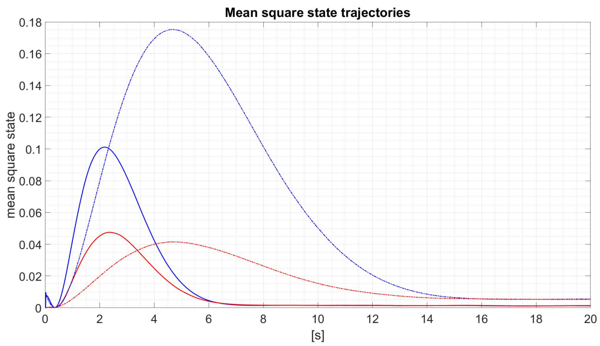


Fig. 5. Mean square state trajectories in closed-loop in
- blue obtained with the Markovian LO;
- red obtained with the Markovian CE;
- dashed blue obtained with the Bernoullian LO;
- dashed red obtained with the Bernoullian CE.

existing between the LO and the CE. The first difference can be individuated in the resulting performance index $J_L^* = 65$ for the LO, and $J_C^* = 43$ for the CE. The performance index shows that the cost achieved by the LO is higher with respect to the one achieved by the CE, see also Remark 18. Moreover, Fig. 4 highlights the behavior of the error trajectories for each observer. After the transient, the error trajectories obtained by the CE become smooth faster with respect to the error trajectories obtained by the LO, which takes 20 samples to become smooth.

IX. Conclusion

This paper presents estimation techniques and detectability conditions for WCNs modeled via MJLSs (under TCP-like communication scheme). The resulting output-feedback control over wireless medium finds applications in industrial automation, telesurgery, smart grids, and intelligent transportation, where communication non-idealities must be considered to guarantee acceptable closed-loop performance.We generalize the results from [4] by using the Markov modeling of the wireless channel and introducing the stabilizability and detectability conditions accounting for the communication link mode, see also Remark 24. As future developments, we plan to investigate the same WCN scenario under a UDP-like communication scheme.

APPENDIX

Technical preliminaries

Since, for finite-dimensional linear spaces, all norms are equivalent, [28, Th. 4.27] from a topological viewpoint, as vector norms we use variants of vector p-norms. For what concerns the matrix norms, we use ℓ_1 and ℓ_2 norms [29, p. 341], which treat $n_r \times n_c$ matrices as vectors of size $n_r n_c$, and use one of the related p-norms. The definition of ℓ_1 and ℓ_2 norms is based on the operation of vectorization of a matrix, $\mathrm{vec}(\cdot)$, which is further used in the definition of the operator $\hat{\varphi}(\cdot)$, to be applied to any block matrix, e.g., $\mathbf{\Phi} = [\Phi_m]_{m=1}^C$: $\hat{\varphi}(\mathbf{\Phi}) \triangleq [\mathrm{vec}(\Phi_1), \dots, \mathrm{vec}(\Phi_C)]'$. The linear operator $\hat{\varphi}(\cdot)$ is a uniform homeomorphism, its inverse operator $\hat{\varphi}^{-1}(\cdot)$ is uniformly continuous [30], and any bounded linear operator in $\mathbb{B}(\mathbb{F}^{Cn_r \times n_c})$ can be represented in $\mathbb{B}(\mathbb{F}^{Cn_r n_c})$ through $\hat{\varphi}(\cdot)$.

A. Mode-dependent estimation techniques

Proof: [**Proposition 1**] Define matrices $\mathbf{\Lambda}_1 \triangleq \mathcal{N}\mathcal{C}$ and $\mathbf{\Lambda}_2 \triangleq \mathcal{N}^*\mathcal{C}^*$, with $\mathcal{C}$ and $\mathcal{N}$ in (23). Then, $\rho(\mathbf{\Lambda}_1) = \rho(\mathbf{\Lambda}_2)$. For all $\mathbf{S} = [S_m]_{m=1}^I$ in $\mathbb{H}^{In_x}$, $\hat{\varphi}(\mathcal{V}(\mathbf{S})) = \mathbf{\Lambda}_1 \hat{\varphi}(\mathbf{S}), \hat{\varphi}(\mathcal{L}(\mathbf{S})) = \mathbf{\Lambda}_2 \hat{\varphi}(\mathbf{S})$, see, e.g., [11, *Rem. 3.5*], and thus, (i) follows. By setting, for all $m \in \mathbb{S}_\eta$, $\widehat{\Gamma}_{m0} = \Gamma_{m0}$, $\widehat{\Gamma}_{m1} = \Gamma_{m1}$, and $\mathbf{\Lambda}_3 \triangleq \mathcal{C}\mathcal{N}$ the following equalities hold: $\rho(\mathbf{\Lambda}_1) = \rho(\mathbf{\Lambda}_2) = \rho(\mathbf{\Lambda}_3)$. Recalling that $\hat{\varphi}(\mathcal{T}(\mathbf{S})) = \mathbf{\Lambda}_3 \hat{\varphi}(\mathbf{S}), \forall \mathbf{S} \in \mathbb{H}^{In_x}$, condition (ii) follows. ■

Proof: [**Proposition 2**] Applying (24)-(25), and (11), the assumptions on noise sequence, the product $GH^* = \mathbb{O}_{n_x}$, (7)-(6), the proof is straightforward, see [19] for more details. ■

Proof: [**Proposition 3**] Consider the error system (15). Recall the definition (28) of $\mathbf{m}(k+1)$, for $k \in \mathbb{N}$. By assumption $(a.3)$, applying the property $\mathbb{E}[w_k] = \mathbf{O}_{n_w}$, the definitions of transition probability and of $\hat{\gamma}_m$, $m \in \mathbb{S}_\eta$, the expression of $\mathbf{m}(k+1)$ in (31) follows. Consider the definition (29) of $\mathbf{Z}(k)$, for $k \in \mathbb{N}$. By applying assumption $(a.3)$, the properties $\mathbb{E}[w_k] = \mathbf{O}_{n_w}, \mathbb{E}[w_k w_k^*] = \mathbb{I}_{n_w}$ in (6), $GH^* = 0$ in (7), definitions of transition probability and of $\hat{\gamma}_m$, the recursive expression of $Z_n(k+1)$, for $m, n \in \mathbb{S}_\eta$, follows. Consider $\mathcal{T}$ and $\widehat{\mathcal{O}}$ defined in (18) and (22), respectively. By setting $\widehat{\Gamma}_{m1} = A + A\widehat{M}_m L$, $\widehat{\Gamma}_{m0} = A$, $\widehat{\mathbf{M}} = [\widehat{M}_m]_{m=1}^I$, for $m \in \mathbb{S}_\eta$, $\boldsymbol{\pi}(k) = [\pi_m(k)]_{m=1}^I$, (31) follows, completing the proof. ■

Proof: [**Proposition 4**] Assume that MJLS (5) is *Strict-MSD*. Then, there exists a mode-dependent filtering gain $\widehat{M}_n \in \mathbb{F}^{n_x \times n_y}$ such that $\rho(\mathcal{T}) < 1$, with $\mathcal{T} \in \mathbb{B}(\mathbb{H}^{In_x})$ in (18), for $\widehat{\Gamma}_{n1} = A + A\widehat{M}_n L$, $\widehat{\Gamma}_{n0} = A$, $n \in \mathbb{S}_\eta$. Pick the filtering gain $\check{M}_n = A\widehat{M}_n \in \mathbb{F}^{n_x \times n_y}$. By setting $\Gamma_{n1} = A + \check{M}_n L$, $\Gamma_{n0} = A$, we have $\widehat{\Gamma}_{n1} = \Gamma_{n1}$ and $\widehat{\Gamma}_{n0} = \Gamma_{n0}$. Consider now the operator $\mathcal{V}$ from (20), for Γ_{n1} and Γ_{n0} defined above. By Proposition 1, $\rho(\mathcal{V}) = \rho(\mathcal{T})$, which implies $\rho(\mathcal{V}) < 1$. Thus, (5) is *MSD*. ■

Proof: [**Theorem 1**] The implication $(i) \implies (ii)$ follows from the Schur complement [11, Lemma 2.23]. On the other hand, assume that (ii) holds. By the optimality of the solution of (35) and the *MSD* of (5), (i) follows. Moreover, the solutions of (35) and (34) coincide, see [19] for more details. ■

Proof: [**Theorem 2**] Assume that $\widehat{\mathbf{Y}} = [\widehat{Y}_n]_{n=1}^I$ is a MS stabilizing solution for filtering CAREs (34), so that system (5) is *MSD*, with $\check{M}_n = \mathcal{M}_n(\widehat{\mathbf{Y}})$, $n \in \mathbb{S}_\eta$. By some technical results from [19], we have the existence of a maximal solution $\mathbf{Y}^+ \in \mathbb{M}$, satisfying $\mathbf{Y}^+ = \mathcal{Y}(\mathbf{Y}^+)$, such that $\widehat{\mathbf{Y}} - \mathbf{Y}^+ \succeq 0$ and $\widehat{\mathbf{Y}} - \mathbf{Y}^+ \preceq 0$. Thus, $\widehat{\mathbf{Y}} = \mathbf{Y}^+$. See [19] for more details. ■

The following result proves the equivalence of the two estimation techniques for the *MSS*.

Proof: [**Lemma 1**] Assume the statement (i) holds. Set $\mathbf{Y}(0) = [Y_n(0)]_{n=1}^I \in \mathbb{H}^{In_x,+}$, $\forall \mathbf{Z}(0) = [Z_n(0)]_{n=1}^I \in \mathbb{H}^{In_x,+}$, as $Y_n(0) = \widehat{\mathcal{A}}_n(\mathbf{Z}(0), \boldsymbol{\pi}(0)) - \tilde{\mathcal{B}}_n(\mathbf{Z}(0), \boldsymbol{\pi}(0))$, with $\tilde{\mathcal{B}}_n(\mathbf{Z}(0), \boldsymbol{\pi}(0)) \triangleq \hat{\gamma}_n \widehat{\mathcal{C}}_n(\mathbf{Z}(0)) \widehat{\mathcal{R}}_n^{-1}(\mathbf{Z}(0), \boldsymbol{\pi}(0)) \widehat{\mathcal{C}}_n^*(\mathbf{Z}(0))$. By (i), the limit for $k \to \infty$ of $\mathbf{Y}(k)$ converges to $\mathbf{Y}$ in $\mathbb{H}^{In_x,+}$ satisfying $\mathbf{Y} = \mathcal{Y}(\mathbf{Y})$. Then, for $n \in \mathbb{S}_\eta$ and $k \in \mathbb{N}$, the following equalities hold: $Y_n(k) = \widehat{\mathcal{A}}_n(\mathbf{Z}(k), \boldsymbol{\pi}(k)) - \tilde{\mathcal{B}}_n(\mathbf{Z}(k), \boldsymbol{\pi}(k))$, with $\tilde{\mathcal{B}}_n(\mathbf{Z}(k), \boldsymbol{\pi}(k)) \triangleq \hat{\gamma}_n \widehat{\mathcal{C}}_n(\mathbf{Z}(k)) \widehat{\mathcal{R}}_n^{-1}(\mathbf{Z}(k), \boldsymbol{\pi}(k)) \widehat{\mathcal{C}}_n^*(\mathbf{Z}(k))$, and $Z_n(k+1) = \mathcal{D}_n(\mathbf{Y}(k))$. This implies that the limit for $k \to \infty$ of $\mathbf{Z}(k)$ converges to $\mathcal{D}(\mathbf{Y}) = \mathcal{Z}(\mathbf{Z})$.
Assume that (ii) holds. Set $\mathbf{Z}(0) = [Z_m(0)]_{m=1}^I \in \mathbb{H}^{In_x,+}$, $\forall\, \mathbf{Y}(0) = [Y_m(0)]_{m=1}^I \in \mathbb{H}^{In_x,+}$, as $\mathbf{Z}(0) = \mathcal{D}(\mathbf{Y}(0))$.

From (ii), $\lim_{k\to\infty}\mathbf{Z}(k)=\mathbf{Z}$, with $\mathbf{Z}=\mathcal{Z}(\mathbf{Z})$. Then, for $m\in\mathbb{S}_\eta$, $Y_m(k+1)=\widehat{\mathcal{A}}_m(\mathbf{Z}(k),\boldsymbol{\pi}(k))-\tilde{\mathcal{B}}_m(\mathbf{Z}(k),\boldsymbol{\pi}(k))$ and $Z_n(k)=\mathcal{D}_n(\mathbf{Y}(k))$, implying $\lim_{k\to\infty}\mathbf{Y}(k)=\mathbf{Y}$, $\mathbf{Y}=\mathcal{Y}(\mathbf{Y})$. Assume that $\mathbf{Y}\in\mathbb{M}$ is the *MS* stabilizing solution of the filtering CAREs $\mathbf{Y}=\mathcal{Y}(\mathbf{Y})$. Then, $\mathcal{M}_n(\mathbf{Y})$ defined by (32) is such that the spectral radius $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ defined in (20) for $\Gamma_{n1}=A+\mathcal{M}_n(\mathbf{Y})L$, $\Gamma_{n0}=A$, and $n\in\mathbb{S}_\eta$. By setting $Z_n=\mathcal{D}_n(\mathbf{Y})$, the following equality holds for any $n\in\mathbb{S}_\eta$: $\mathcal{M}_n(\mathbf{Y})=A\widehat{\mathcal{M}}_n(\mathbf{Z})$. Considering $\widehat{\Gamma}_{n1}=A+A\widehat{\mathcal{M}}_n(\mathbf{Z})L$ and $\widehat{\Gamma}_{n0}=A$, we obtain $\widehat{\Gamma}_{n1}=\Gamma_{n1}$ and $\widehat{\Gamma}_{n0}=\Gamma_{n0}$. By Proposition 1, $\rho(\mathcal{V})=\rho(\mathcal{T})$, and, consequently, $\rho(\mathcal{T})<1$. Moreover, the optimal performance index achieved by the CE is $J_C^*=\sum_{n=1}^{I}\operatorname{tr}(Z_n)$. ■

In the following, all mathematical preliminaries and motivations leading to the separation principle are illustrated concerning the output-feedback controller designed with the Markovian LO. A reduced version of the proof is reported in [19]. Define, for $k\in\mathbb{N}$, $\ell,i,j\in\mathbb{S}_\theta$, $w_{\ell i}(k)\triangleq\mathbb{E}[x_k\mathbf{1}_{\{\theta_{k-1}=\ell,\theta_k=i\}}]$, $W_{\ell i}(k)\triangleq\mathbb{E}[x_kx_k^*\mathbf{1}_{\{\theta_{k-1}=\ell,\theta_k=i\}}]$ and set $\mathbf{w}(k)\triangleq[w_{\ell i}(k)]_{\ell,i=1}^N$, $\mathbf{W}(k)\triangleq[W_{\ell i}(k)]_{\ell,i=1}^N$. For $\mathbf{V}=[V_{ij}]_{i,j=1}^N\in\mathbb{F}^{Nn_x\times Nn_x}$, define the operator $\widehat{\mathcal{L}}(\cdot)=[\widehat{\mathcal{L}}_{ij}(\cdot)]_{i,j=1}^N\in\mathbb{B}(\mathbb{F}^{Nn_x\times Nn_x})$, with

$$\widehat{\mathcal{L}}_{ij}(\mathbf{V})\triangleq\Big\{A\sum_{l=1}^N V_{li}A^*+\hat{\nu}_iB\sum_{l=1}^N F_lV_{li}F_l^*B^*+\hat{\nu}_iB\sum_{l=1}^N F_lV_{li}A^*+\hat{\nu}_iA\sum_{l=1}^N V_{li}F_l^*B^*\Big\}p_{ij}. \quad (44)$$

For $\mathbf{Y}=[Y_m]_{m=1}^I\in\mathbb{H}^{In_x}$, $\boldsymbol{\beta}=[\beta_{\ell i}]_{\ell,i=1}^N\in\mathbb{R}^{N\times N}$, $i,j\in\mathbb{S}_\theta$, define the operator $\mathcal{H}(\cdot,\cdot):\mathbb{H}^{In_x}\times\mathbb{R}^{N\times N}\to\mathbb{F}^{Nn_x\times Nn_x}$ as $\mathcal{H}(\mathbf{Y},\boldsymbol{\beta})\triangleq[\mathcal{H}_{ij}(\mathbf{Y},\boldsymbol{\beta})]_{i,j=1}^N$, with

$$\mathcal{H}_{ij}(\mathbf{Y},\boldsymbol{\beta})\triangleq\hat{\nu}_iB\sum_{\ell=1}^N\beta_{\ell i}F_\ell\Big(\sum_{m=1}^I Y_m\Big)F_\ell^*B^*p_{ij}+GG^*\beta_{ij}.$$

Proposition 8: Consider the MJLS (5) and the closed-loop system dynamics (37). Then, $\forall\ k\in\mathbb{N},\ i,j\in\mathbb{S}_\theta$,

$$W_{ij}(k+1)=\widehat{\mathcal{L}}_{ij}(\mathbf{W}(k))+\mathcal{H}_{ij}(\check{\mathbf{Y}}(k),\tilde{\boldsymbol{\varpi}}(k))-2\Re\Big\{\hat{\nu}_iB\sum_{\ell=1}^N F_\ell\Big(\sum_{m=1}^I\check{m}_m(k)\Big)w_{\ell i}^*(k)A^*+\hat{\nu}_iB\sum_{\ell=1}^N F_\ell\Big(\sum_{m=1}^I\check{m}_m(k)\Big)w_{\ell i}^*(k)F_\ell^*B^*\Big\}p_{ij}. \quad (45)$$

Proof: From (37) and (11), recalling the definition of $\widehat{\mathcal{L}}$ in (44), by assumptions $(a.2)-(a.3)$, applying (6) and the independence of sequences θ_k and $\check{e}_k$,

$$W_{ij}(k+1)=\widehat{\mathcal{L}}_{ij}(\mathbf{W}(k))+\Big(\hat{\nu}_iB\sum_{\ell=1}^N\tilde{\varpi}_{\ell i}(k)F_\ell\mathbb{E}[\check{e}_k\check{e}_k^*]F_\ell^*B^*+GG^*\sum_{\ell=1}^N\tilde{\varpi}_{\ell i}(k)-\hat{\nu}_i2\Re\Big(B\sum_{\ell=1}^N F_\ell\mathbb{E}[\check{e}_k]w_{\ell i}^*(k)A^*+B\sum_{\ell=1}^N F_\ell\mathbb{E}[\check{e}_k]w_{\ell i}^*(k)F_\ell^*B^*\Big)\Big)p_{ij},$$

and thus, equation (45) follows. The proof is complete. ■

Proof: [**Theorem 3**] Assume that (ii) holds. Then, by Definition 2, there exists a mode-dependent control gain F_ℓ, $\ell\in\mathbb{S}_\theta$, that makes the dynamics of x_k *MSS*. Consequently, by [13, Prop. 3], $\rho(\widehat{\mathcal{L}})<1$, with $\widehat{\mathcal{L}}$ in (44). By Definition 4, there exists a mode-dependent filtering gain $\check{M}_n$, $n\in\mathbb{S}_\eta$, such that $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (20), for $\Gamma_{n1}=A+\check{M}_nL$ and $\Gamma_{n0}=A$. By Proposition 2,

$$\check{\mathbf{Y}}(k+1)=\mathcal{V}(\check{\mathbf{Y}}(k))+\mathcal{O}(\check{\mathbf{M}},\boldsymbol{\pi}(k)). \quad (46)$$

Since $\rho(\mathcal{V})<1$, by $(a.4)$, from (46), $\lim_{k\to\infty}\check{\mathbf{Y}}(k)=\check{\mathbf{Y}}$,

$$\check{\mathbf{Y}}=\mathcal{V}(\check{\mathbf{Y}})+\mathcal{O}(\check{\mathbf{M}},\boldsymbol{\pi}^\infty), \quad (47)$$

and thus, for $i,j\in\mathbb{S}_\theta$, $\lim_{k\to\infty}\mathcal{H}_{ij}(\check{\mathbf{Y}}(k),\tilde{\boldsymbol{\varpi}}(k))=\mathcal{H}_{ij}(\check{\mathbf{Y}},\tilde{\boldsymbol{\varpi}}^\infty)$. From (27) and (47), by [11, Propositions 3.6 and 3.36], we obtain $\lim_{k\to\infty}\check{\mathbf{m}}(k)=\mathbf{O}_{In_x}$. By Proposition 8, for $i,j\in\mathbb{S}_\theta$, $\lim_{k\to\infty}W_{ij}(k+1)=\widehat{\mathcal{L}}_{ij}(\mathbf{W})+\mathcal{H}_{ij}(\check{\mathbf{Y}},\tilde{\boldsymbol{\varpi}}^\infty)$, and thus, there exists $\mathbf{W}=[W_{ij}]_{i,j=1}^N$, with $W_{ij}\in\mathbb{F}_+^{n_x\times n_x}$ satisfying $W_{ij}=\lim_{k\to\infty}W_{ij}(k)$. Moreover, by [13, Prop. 2], $\lim_{k\to\infty}w_{\ell i}(k)=w_{\ell i}\in\mathbb{F}^{n_x}$, $\ell,i\in\mathbb{S}_\theta$. Therefore, the closed-loop system is *MSS* by Definition 1, implying (i). To prove the converse of the theorem, assume now that (i) holds. Then, there exists $\mathbf{W}=[W_{ij}]_{i,j=1}^N$, with $W_{ij}=\lim_{k\to\infty}W_{ij}(k)$. By Proposition 8, W_{ij}, $i,j\in\mathbb{S}_\theta$, can be written as follows. $W_{ij}=\lim_{k\to\infty}W_{ij}(k+1)$, with $W_{ij}(k+1)$ in (45). Thus, there exists $\check{\mathbf{Y}}\in\mathbb{H}^{In_x,+}$, such that $\lim_{k\to\infty}\check{\mathbf{Y}}(k)=\check{\mathbf{Y}}$, with $\check{\mathbf{Y}}$ satisfying (47). Therefore, the error system (11) is *MSS*. By [11, Th. 3.33, Th. 3.9], we have that condition $(ii\text{-}a)$ holds. Moreover, by [11, Propositions 3.6 and 3.36], $\lim_{k\to\infty}\check{\mathbf{m}}(k)=\mathbf{O}_{In_x}$, and thus, the following equality holds for $i,j\in\mathbb{S}_\theta$, $W_{ij}=\widehat{\mathcal{L}}_{ij}(\mathbf{W})+\mathcal{H}_{ij}(\check{\mathbf{Y}},\tilde{\boldsymbol{\varpi}}^\infty)$, implying that the mode-dependent control gain F_ℓ, $\ell\in\mathbb{S}_\theta$, stabilizes the dynamics (37) in the *MS* sense, i.e., condition $(ii\text{-}b)$ holds. ■

The detailed proof of the separation principle concerning the OFC based on the CE is presented in the following. Define $\widehat{\mathcal{H}}(\cdot,\cdot,\cdot):\mathbb{H}^{In_x}\times\mathbb{R}^{N\times N}\times\mathbb{R}^I\to\mathbb{F}^{Nn_x\times Nn_x}$, for $\mathbf{Z}=[Z_m]_{m=1}^I\in\mathbb{H}^{In_x}$, $\boldsymbol{\beta}=[\beta_{\ell i}]_{\ell,i=1}^N\in\mathbb{R}^{N\times N}$, $\boldsymbol{\sigma}=[\sigma_m]_{m=1}^I\in\mathbb{R}^I$, $i,j\in\mathbb{S}_\theta$, as $\widehat{\mathcal{H}}(\mathbf{Z},\boldsymbol{\beta},\boldsymbol{\sigma})\triangleq[\widehat{\mathcal{H}}_{ij}(\mathbf{Z},\boldsymbol{\beta},\boldsymbol{\sigma})]_{i,j=1}^N$, with

$$\widehat{\mathcal{H}}_{ij}(\mathbf{Z},\boldsymbol{\beta},\boldsymbol{\sigma})\triangleq\sum_{\ell=1}^N\sum_{m=1}^I\Big\{\beta_{\ell i}\hat{\nu}_iBF_\ell Z_mF_\ell^*B^*+\beta_{\ell i}\hat{\nu}_i\hat{\gamma}_mBF_\ell\widehat{M}_mLZ_mL^*\widehat{M}_m^*F_\ell^*B^*+\beta_{\ell i}\hat{\nu}_i\hat{\gamma}_mBF_\ell Z_mL^*\widehat{M}_m^*F_\ell^*B^*+\beta_{\ell i}\hat{\nu}_i\hat{\gamma}_mBF_\ell\widehat{M}_mLZ_mF_\ell^*B^*+\beta_{\ell i}\sigma_mGG^*+\beta_{\ell i}\sigma_m\hat{\nu}_i\hat{\gamma}_mBF_\ell\widehat{M}_mHH^*\widehat{M}_m^*F_\ell^*B^*\Big\}p_{ij}. \quad (48)$$

Proposition 9: Consider the MJLS (5) and the closed-loop system dynamics (39). Then, $\forall\ k\in\mathbb{N},\ i,j\in\mathbb{S}_\theta$,

$$W_{ij}(k+1)=\widehat{\mathcal{L}}_{ij}(\mathbf{W}(k))+\widehat{\mathcal{H}}_{ij}(\mathbf{Z}(k),\tilde{\boldsymbol{\varpi}}(k),\boldsymbol{\pi}(k))+2\Re\Big(\sum_{\ell=1}^N\sum_{m=1}^I\{\hat{\nu}_iBF_\ell m_m(k)w_{\ell i}^*(k)A^*+\hat{\nu}_i\hat{\gamma}_mBF_\ell\widehat{M}_mLm_m(k)w_{\ell i}^*(k)A^*+\hat{\nu}_iBF_\ell m_m(k)w_{\ell i}^*(k)F_\ell^*B^*+\hat{\nu}_i\hat{\gamma}_mBF_\ell\widehat{M}_mLm_m(k)w_{\ell i}^*(k)F_\ell^*B^*\}\Big)p_{ij}, \quad (49)$$

where $\widehat{\mathcal{L}}_{ij}$ and $\widehat{\mathcal{H}}_{ij}$ are defined in (44) and (48), respectively.

Proof: From (39) and (15), recalling $\widehat{\mathcal{L}}$ operator in (44), by assumptions $(a.2)-(a.3)$, applying the properties $\mathbb{E}[w_k]=\mathbf{O}_{n_w}$, $\mathbb{E}[w_kw_k^*]=\mathbb{I}_{n_w}$, in (6), the definition of transition probability, the independence of sequences $\{e_k\}$ and $\{\theta_k\}$ (see Remark 12), equation (49) follows. ■

Proof: [**Theorem 4**] Assume (ii) holds. Then, there exists a mode-dependent control gain F_ℓ, $\ell\in\mathbb{S}_\theta$, that makes the dynamics of x_k *MSS*. Consequently, by [13, Prop. 3], $\rho(\widehat{\mathcal{L}})<1$, with $\widehat{\mathcal{L}}$ in (44). By Definition 5, there exists a mode-dependent filtering gain $\widehat{M}_n$, $n\in\mathbb{S}_\eta$, such that $\rho(\mathcal{T})<1$, with $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (18), $\widehat{\Gamma}_{n1}=A+A\widehat{M}_nL$ and $\widehat{\Gamma}_{n0}=A$. By Proposition 3,

$\mathbf{Z}(k+1)=\mathcal{T}(\mathbf{Z}(k))+\widehat{\mathcal{O}}(\widehat{\mathbf{M}},\boldsymbol{\pi}(k))$. Thus, from $\rho(\mathcal{T})<1$ and the assumption $(a.4)$, $\lim_{k\to\infty}\mathbf{Z}(k)=\mathbf{Z}\in\mathbb{H}^{In_x,+}$, with

$$\mathbf{Z}=\mathcal{T}(\mathbf{Z})+\widehat{\mathcal{O}}(\widehat{\mathbf{M}},\boldsymbol{\pi}^{\infty}). \tag{50}$$

Therefore, $\lim_{k\to\infty}\widehat{\mathcal{H}}_{ij}(\mathbf{Z}(k),\tilde{\boldsymbol{\varpi}}(k),\boldsymbol{\pi}(k))=\widehat{\mathcal{H}}_{ij}(\mathbf{Z},\tilde{\boldsymbol{\varpi}}^{\infty},\boldsymbol{\pi}^{\infty})$, $i,j\in\mathbb{S}_\theta$. From (31) and (50), by [11, Propositions 3.6 and 3.36], $\lim_{k\to\infty}\mathbf{m}(k)=\mathbf{O}_{In_x}$. By Proposition 9, $\lim_{k\to\infty}W_{ij}(k+1)=\widehat{\mathcal{L}}_{ij}(\mathbf{W})+\widehat{\mathcal{H}}_{ij}(\mathbf{Z},\tilde{\boldsymbol{\varpi}}^{\infty},\boldsymbol{\pi}^{\infty})$, $i,j\in\mathbb{S}_\theta$. Thus, there exists $\mathbf{W}=[W_{ij}]_{i,j=1}^{N}$, with $W_{ij}\in\mathbb{F}_+^{n_x\times n_x}$, satisfying, for $i,j\in\mathbb{S}_\theta$, $W_{ij}=\lim_{k\to\infty}W_{ij}(k+1)$. Moreover, by [13, Prop. 2], we get $\lim_{k\to\infty}w_{\ell i}(k)=w_{\ell i}\in\mathbb{F}^{n_x}$, $\ell,i\in\mathbb{S}_\theta$. Therefore, the closed-loop system is *MSS*, i.e., (i) holds.

To prove the converse statement, assume (i) holds. Then, there exists $\mathbf{W}=[W_{ij}]_{i,j=1}^{N}$, with $W_{ij}=\lim_{k\to\infty}W_{ij}(k)$. By Proposition 9, W_{ij}, $i,j\in\mathbb{S}_\theta$, satisfies (49), with $\mathbf{Z}(k)$ satisfying (31). This implies that there exists $\mathbf{Z}\in\mathbb{H}^{In_x,+}$, such that $\lim_{k\to\infty}\mathbf{Z}(k)=\mathbf{Z}$, with $\mathbf{Z}$ satisfying (50). Therefore, the error system (15) is *MSS*, and, by [11, Th. 3.33, Th. 3.9], condition $(ii\text{-}a)$ holds. Moreover, by [11, Propositions 3.6 and 3.36], we have that $\lim_{k\to\infty}\mathbf{m}(k)=\mathbf{O}_{In_x}$. Thus, the following equality holds for $i,j\in\mathbb{S}_\theta$: $W_{ij}=\widehat{\mathcal{L}}_{ij}(\mathbf{W})+\widehat{\mathcal{H}}_{ij}(\mathbf{Z},\tilde{\boldsymbol{\varpi}}^{\infty},\boldsymbol{\pi}^{\infty})$, implying that the mode-dependent control gain F_ℓ, $\ell\in\mathbb{S}_\theta$, makes the dynamics (39) *MSS*, i.e., condition $(ii\text{-}b)$ holds. ∎

B. Proofs for the mode-independent output-feedback

This section reports the results on the mode-independent OFC.

Proof: [**Proposition 5**] Consider the MJLS (5). Assume that (5) is *Strong-MSD*. Then, there exists a mode-independent filtering gain $\check{M}^b\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (20), for $\Gamma_{n1}=A+\check{M}^bL$, $\Gamma_{n0}=A$ and $n\in\mathbb{S}_\eta$. Pick the mode-dependent filtering gain $\check{M}_n=\check{M}^b$ and consider $\mathcal{V}$ with $\Gamma_{n1}=A+\check{M}_nL$ and $\Gamma_{n0}=A$. The condition $\rho(\mathcal{V})<1$ is again satisfied, and implication (i) holds. Assume now that (5) is *Strong-Strict-MSD*. Then, there exists a mode-independent filtering gain $\widehat{M}^b\in\mathbb{F}^{n_x\times n_y}$, such that $\rho(\mathcal{T})<1$, with $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (18), for $\widehat{\Gamma}_{n1}=A+A\widehat{M}^bL$, $\widehat{\Gamma}_{n0}=A$ and $n\in\mathbb{S}_\eta$. For all $m\in\mathbb{S}_\eta$, pick the mode-dependent filtering gain $\widehat{M}_m=\widehat{M}^b$ and consider $\mathcal{T}$ with $\widehat{\Gamma}_{m1}=A+A\widehat{M}_mL$ and $\widehat{\Gamma}_{m0}=A$. The condition $\rho(\mathcal{T})<1$ is again satisfied, and the implication *Strong-Strict-MSD* $\Longrightarrow$ *Strict-MSD* holds. Moreover, if (5) is *Strong-Strict-MSD*, then, there exists a mode-independent filtering gain $\check{M}^b=A\widehat{M}^b$, such that, setting $\Gamma_{n1}=A+\check{M}^bL$ and $\Gamma_{n0}=A$, the following equalities are satisfied: $\widehat{\Gamma}_{n0}=\Gamma_{n0}$, $\widehat{\Gamma}_{n1}=\Gamma_{n1}$, for all $n\in\mathbb{S}_\eta$. Consider now $\mathcal{V}$, with Γ_{n1}, Γ_{n0} defined above. By Proposition 1, $\rho(\mathcal{V})=\rho(\mathcal{T})$, which implies $\rho(\mathcal{V})<1$, and thus, *Strong-Strict-MSD* $\Longrightarrow$ *Strong-MSD*. ∎

Proof: [**Theorem 5**] Condition (ii) follows from [13, Th. 3], so we only need to prove condition (i). By assumption, $\boldsymbol{\pi}^\infty=[\pi_m^\infty]_{m=1}^I$ is the stationary distribution of the SL modes, and thus, $\pi_n^\infty=\sum_{m=1}^I q_{mn}\pi_m^\infty$. Moreover, the probability $\mathring{\gamma}=\sum_{n=1}^I\pi_n^\infty\hat{\gamma}_n$ can be written as $\mathring{\gamma}=\sum_{m=1}^I\pi_m^\infty\sum_{n=1}^I q_{mn}\hat{\gamma}_n$. By applying the property $\sum_{m=1}^I\pi_m^\infty=1$, the filtering MARE (40) can be rewritten as $\sum_{m=1}^I\pi_m^\infty\{Y_\infty^b-\mathring{\mathcal{A}}^b(Y_\infty^b)+\mathring{\mathcal{B}}_m^b(Y_\infty^b)\}=0$, with $\mathring{\mathcal{B}}_m^b(Y_\infty^b)\triangleq\sum_{n=1}^I q_{mn}\hat{\gamma}_n(\mathring{\mathcal{C}}^b(Y_\infty^b)\mathring{\mathcal{R}}^b(Y_\infty^b)^{-1}\mathring{\mathcal{C}}^{b*}(Y_\infty^b))$, holding for all $\boldsymbol{\pi}=[\pi_m]_{m=1}^I$, if and only if, for all $m\le I$,

$$Y_\infty^b=\mathring{\mathcal{A}}^b\left(Y_\infty^b\right)-\zeta_m\mathring{\mathcal{C}}^b\left(Y_\infty^b\right)\mathring{\mathcal{R}}^b\left(Y_\infty^b\right)^{-1}\mathring{\mathcal{C}}^{b*}\left(Y_\infty^b\right), \tag{51}$$

with $\zeta_m=\sum_{n=1}^I q_{mn}\hat{\gamma}_n$. Equation (51) is exactly equation (40), where, as required by the mode-independence, $Y_m=Y_\infty^b$, for all $m\in\mathbb{S}_\eta$. This completes the proof. ∎

Technical results concerning *MSD* are proved in the following.

Proof: [**Proposition 6**] Assume (i) holds. Then, there exists a mode-dependent filtering gain $\check{M}_n\in\mathbb{F}^{n_x\times n_y}$, $n\in\mathbb{S}_\eta$, such that $\rho(\mathcal{V})<1$, with $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (20), for $\Gamma_{n1}=A+\check{M}_nL$ and $\Gamma_{n0}=A$. Consider the operator $\mathcal{L}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (19), with the same Γ_{n1} and Γ_{n0} given for $\mathcal{V}$. By Proposition 1, $\rho(\mathcal{V})=\rho(\mathcal{L})$, and, therefore, $\rho(\mathcal{L})<1$. By applying [11, Th. 3.9], we have that, for $\mathbf{V}=[V_m]_{m=1}^I\in\mathbb{H}^{In_x,+}$, $V_m\succ 0$, $m\in\mathbb{S}_\eta$, $\mathcal{L}_m(\mathbf{V})-V_m\prec 0$. Taking $\mathbf{W}_1=\mathbf{V}$ and $Z_m=\mathcal{E}_m(\mathbf{W}_1)$, condition (42c) is satisfied. By choosing $W_{m2}=Z_m\check{M}_m$, $W_{m3}=W_{m2}^*Z_m^{-1}W_{m2}$ and substituting these expressions in $\mathcal{L}_m(\mathbf{V})-V_m$, condition (42a) follows. Recalling that $W_{m3}\succeq W_{m2}^*Z_m^{-1}W_{m2}$, by the Schur complement [11, Lemma 2.23], we get (42b). Thus, statement $(i)\implies(iii)$. Let us prove that $(iii)\implies(i)$. Assume that (iii) holds and choose the filtering gain as $\check{M}_m=Z_m^{-1}W_{m2}$. Consider again the operator $\mathcal{L}$ defined above. From (42b), by the Schur complement, we have $W_{m3}\succeq W_{m2}^*Z_m^{-1}W_{m2}$, and, from (42c), $Z_m\succeq\mathcal{E}_m(\mathbf{W}_1)$. Thus, by condition (42a), we get $\mathcal{L}_m(\mathbf{W}_1)-W_{m1}\prec 0$. By [11, Th. 3.9], $\rho(\mathcal{L})<1$. Consider the operator $\mathcal{V}$ in (20), for $\Gamma_{m1}=A+\check{M}_mL$, $\Gamma_{m0}=A$, and $m\in\mathbb{S}_\eta$. By Proposition 1: $\rho(\mathcal{V})=\rho(\mathcal{L})$, and, consequently, $\rho(\mathcal{V})<1$. Thus, the system (5) is *MSD*, and statement (i) holds.

Assume (ii) holds. Then, there exists a mode-dependent filtering gain $\widehat{M}_n\in\mathbb{F}^{n_x\times n_y}$, $n\in\mathbb{S}_\eta$, such that $\rho(\mathcal{T})<1$, with $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (18), for $\widehat{\Gamma}_{n1}=A+A\widehat{M}_nL$ and $\widehat{\Gamma}_{n0}=A$, $n\in\mathbb{S}_\eta$. Consider the operator $\mathcal{L}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (19), with $\Gamma_{n1}=\widehat{\Gamma}_{n1}$ and $\Gamma_{n0}=\widehat{\Gamma}_{n0}$, for all $n\in\mathbb{S}_\eta$. By Proposition 1, $\rho(\mathcal{T})=\rho(\mathcal{L})$, and, therefore, $\rho(\mathcal{L})<1$. By applying [11, Th. 3.9], we have that, for $\mathbf{V}=[V_m]_{m=1}^I\in\mathbb{H}^{In_x,+}$, $V_m\succ 0$, $m\in\mathbb{S}_\eta$, $\mathcal{L}_m(\mathbf{V})-V_m\prec 0$. By taking $\mathbf{W}_1=\mathbf{V}$, $Z_m=\mathcal{E}_m(\mathbf{W}_1)$, condition (42c) is satisfied. By choosing $W_{m2}=Z_mA\widehat{M}_m$, $W_{m3}=W_{m2}^*Z_m^{-1}W_{m2}$, and substituting these expressions in $\mathcal{L}_m(\mathbf{V})-V_m$, condition (42a) follows. Recalling that $W_{m3}\succeq W_{m2}^*Z_m^{-1}W_{m2}$, by the Schur complement condition (42b) is satisfied, and (iii) holds.

Let us prove that if the matrix A is non-singular, the converse implication, i.e., $(iii)\implies(ii)$, is true. Assume (iii) holds and matrix A is non-singular. Then, the filtering gain can be chosen as $\widehat{M}_m=A^{-1}Z_m^{-1}W_{m2}$. Consider again the operator $\mathcal{L}$ defined above. From (42b), by the Schur complement (see [11, Lemma 2.23]), we have $W_{m3}\succeq W_{m2}^*Z_m^{-1}W_{m2}$ and, from (42c), $Z_m\succeq\mathcal{E}_m(\mathbf{W}_1)$. Thus, by condition (42a), we get $\mathcal{L}_m(\mathbf{W}_1)-W_{m1}\prec 0$ and, by [11, Th. 3.9], $\rho(\mathcal{L})<1$. Consider the operator $\mathcal{T}$ in (18), with $\widehat{\Gamma}_{n1}=A+A\widehat{M}_nL$, $\widehat{\Gamma}_{n0}=A$, and $n\in\mathbb{S}_\eta$. This implies that $\widehat{\Gamma}_{n0}=\Gamma_{n0}$, $\widehat{\Gamma}_{n1}=\Gamma_{n1}$, for all $n\in\mathbb{S}_\eta$, and, by Proposition 1, $\rho(\mathcal{T})=\rho(\mathcal{L})$. Therefore, $\rho(\mathcal{T})<1$, and condition (ii) holds. ∎

Proof: [**Proposition 7**] Assume that (43) holds for some $\mathbf{W}_1=[W_{m1}]_{m=1}^I\in\mathbb{H}^{In_x,+}$, $Z\in\mathbb{F}_+^{n_x\times n_x}$, $W_2\in\mathbb{F}^{n_x\times n_y}$, $W_3\in\mathbb{F}_+^{n_y\times n_y}$, for all $m\in\mathbb{S}_\eta$. Choose the filtering gain as $\check{M}^b=Z^{-1}W_2$ and consider $\mathcal{L}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (19), for $\Gamma_{m1}=A+\check{M}^bL$, $\Gamma_{m0}=A$, and $m\in\mathbb{S}_\eta$. From (43b), by the Schur complement, (see [11, Lemma 2.23]), we have that $W_3\succeq W_2^*Z^{-1}W_2$ and, from (43c), $\mathcal{E}_m(\mathbf{W}_1)\preceq Z$, for all $m\in\mathbb{S}_\eta$.

Thus, by condition (43a), we obtain $\mathcal{L}_m(\mathbf{W}_1)-W_{m1}\prec 0$. By [11, Th. 3.9], it follows that $\rho(\mathcal{L})<1$. Consider the operator $\mathcal{V}\in\mathbb{B}(\mathbb{H}^{In_x})$ defined in (20), with $\Gamma_{m1}=A+\check{M}^bL$, $\Gamma_{m0}=A$, and $m\in\mathbb{S}_\eta$. By Proposition 1, $\rho(\mathcal{V})=\rho(\mathcal{L})$, and, consequently, $\rho(\mathcal{V})<1$. Therefore, the MJLS (5) is *Strong-MSD*, and, thus, the proof of the implication $(i) \implies (ii)$ is complete.

Now, assume conditions (43) hold for some $Z\in\mathbb{F}_+^{n_x\times n_x}$, $\mathbf{W}_1=[W_{m1}]_{m=1}^{I}\in\mathbb{H}^{In_x,+}$, $W_2\in\mathbb{F}^{n_x\times n_y}$, and $W_3\in\mathbb{F}_+^{n_y\times n_y}$. Recall that the matrix A is non-singular by assumption and choose the filtering gain as $\widehat{M}^b=A^{-1}Z^{-1}W_2$. Consider the operator $\mathcal{L}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (19) with $\Gamma_{m1}=A+A\widehat{M}^bL$, $\Gamma_{m0}=A$, and $m\in\mathbb{S}_\eta$. From (43b), by the Schur complement we have that $W_3\succeq W_2^*Z^{-1}W_2$, and from (43c) we have $\mathcal{E}_m(\mathbf{W}_1)\preceq Z$, for all $m\in\mathbb{S}_\eta$. Thus, by condition (43a), we obtain $\mathcal{L}_m(\mathbf{W}_1)-W_{m1}\prec 0$. By [11, Th. 3.9], $\rho(\mathcal{L})<1$. Consider $\mathcal{T}\in\mathbb{B}(\mathbb{H}^{In_x})$ in (18), for $\widehat{\Gamma}_{m1}=A+A\widehat{M}^bL$, $\widehat{\Gamma}_{m0}=A$, and $m\in\mathbb{S}_\eta$. This implies $\widehat{\Gamma}_{m0}=\Gamma_{m0}$ and $\widehat{\Gamma}_{m1}=\Gamma_{m1}$. By Proposition 1, $\rho(\mathcal{L})=\rho(\mathcal{T})$, and, consequently, $\rho(\mathcal{T})<1$. Thus, the system (5) is *Strong-Strict-MSD*. ■

References

[1] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[2] W. P. M. H. Heemels, A. R. Teel, N. van de Wouw, and D. Nesic, "Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1781–1796, 2010.

[3] B. Ding, "Stabilization of linear systems over networks with bounded packet loss and its use in model predictive control," *Automatica*, vol. 47, no. 11, pp. 2526–2533, 2011.

[4] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of Control and Estimation Over Lossy Networks," *Proc. IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[5] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[6] V. Gupta, A. F. Dana, J. P. Hespanha, R. M. Murray, and B. Hassibi, "Data transmission over networks for estimation and control," *IEEE Trans. Autom. Control*, vol. 54, no. 8, pp. 1807–1819, 2009.

[7] M. Pajic, S. Sundaram, G. J. Pappas, and R. Mangharam, "The wireless control network: a new approach for control over networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2305–2318, 2011.

[8] P. Sadeghi, R. A. Kennedy, P. B. Rapajic, and R. Shams, "Finite-state Markov modeling of fading channels - a survey of principles and applications," *IEEE Signal Process. Mag.*, vol. 25, no. 5, pp. 57–80, 2008.

[9] A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel, "Markov jump linear systems and filtering through network transmitted measurements," *Signal Process.*, vol. 90, no. 10, pp. 2842–2850, 2010.

[10] I. Matei, N. C. Martins, and J. S. Baras, "Optimal Linear Quadratic Regulator for Markovian Jump Linear Systems, in the presence of one time-step delayed mode observations," *IFAC Proc.*, vol. 41, no. 2, pp. 8056–8061, 2008.

[11] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, "Discrete-time Markov jump linear systems," Springer, New York, 2005.

[12] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss," *Automatica*, vol. 43, no. 1, pp. 80–87, 2007.

[13] Y. Zacchia Lun and A. D'Innocenzo, "Stabilizability of Markov jump linear systems modeling wireless networked control scenarios," in *IEEE Conf. Decis. Control (CDC)*, 2019, pp. 5766–5772.

[14] A. N. Vargas, L. Acho, G. Pujol, E. F. Costa, J. Y. Hishihara, and J. B. R. do Val, "Output feedback of Markov jump linear systems with no mode observation: An automotive throttle application," *Int. J. Robust Nonlinear Control*, vol. 26, pp. 1980–1993, 2016.

[15] H. J. Chizeck, A. S. Willsky, and D. Castañón, "Discrete-time Markovian-jump linear quadratic optimal control," *Int. J. Control*, vol. 43, no. 1, pp. 213–231, 1986.

[16] Y. Mo, E. Garone, and B. Sinopoli, "LQG control with Markovian packet loss," in *Eur. Control Conf. (ECC)*, 2013, pp. 2380–2385.

[17] Y. Ji and H. J. Chizeck, "Jump linear quadratic gaussian control : Steady-state solution and testable conditions," 1990.

[18] G. F. Franklin, M. L. Workman, and D. Powell, *Digital Control of Dynamic Systems*, 3rd ed. Addison-Wesley, 1997.

[19] A. Impicciatore, Y. Zacchia Lun, P. Pepe, and A. D'Innocenzo, "Optimal output-feedback control and separation principle for Markov jump linear systems modeling wireless networked control scenarios," in *Amer. Control Conf. (ACC)*, 2021, pp. 2700–2706.

[20] R. E. Megginson, *An Introduction to Banach Space Theory*, ser. Graduate Texts in Mathematics. Springer, 1998, vol. 183.

[21] J. W. Brewer, "Kronecker products and matrix calculus in system theory," *IEEE Trans. Circuits Syst.*, vol. 25, no. 9, pp. 772–781, 1978.

[22] Y. Zacchia Lun, C. Rinaldi, A. Alrish, A. D'Innocenzo, and F. Santucci, "On the impact of accurate radio link modeling on the performance of WirelessHART control networks," in *IEEE Conf. Comput. Commun. (INFOCOM)*, 2020, pp. 2430–2439.

[23] C. Luo, J. Ji, Q. Wang, X. Chen, and P. Li, "Channel state information prediction for 5G wireless communications: A deep learning approach," *IEEE Trans. Netw. Sci. Eng.*, vol. 7, no. 1, pp. 227–236, 2018.

[24] Y. Zacchia Lun, A. D'Innocenzo, and M. D. Di Benedetto, "Robust stability of polytopic time-inhomogeneous markov jump linear systems," *Automatica*, vol. 105, pp. 286–297, 2019.

[25] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, 2004.

[26] M. Z. Chen, L. Zhang, H. Su, and G. Chen, "Stabilizing solution and parameter dependence of modified algebraic Riccati equation with application to discrete-time network synchronization," *IEEE Trans. Autom. Control*, vol. 61, no. 1, pp. 228–233, 2016.

[27] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback control of dynamic systems*, 6th ed. Prentice Hall, 2009.

[28] C. S. Kubrusly, *Elements of operator theory*. Birkhäuser, 2001.

[29] R. A. Horn and C. R. Johnson, *Matrix analysis*, 2nd ed. CUP, 2012.

[30] A. W. Naylor and G. R. Sell, *Linear operator theory in engineering and science*, ser. Appl. Math. Sci. Springer, 2000, vol. 40.

Anastasia Impicciatore received the Ph.D. in computer science in 2023, from the University of L'Aquila, Italy. Her research focuses on Wireless Control Networks. Since July 2023, she is a Software Engineer with application on Combat Systems at Leonardo Spa, Rome, Italy.

Yuriy Zacchia Lun is an Assistant Professor at the University of L'Aquila, Italy. His research focuses on automatic control of stochastic hybrid systems and security in cyber–physical domain. He received his Ph.D. in computer science from the Gran Sasso Science Institute, Italy, in 2017. For more information: `zacchialun.com`.

Pierdomenico Pepe received the Laurea degree in electronic engineering (summa cum laude) from the University of Ancona, Ancona, Italy, in 1990, and the Ph.D. degree in electronic engineering in 1996 from the University of L'Aquila, L'Aquila, Italy, where he is currently serving as a Full Professor. His research include stability and nonlinear control theory, with special emphasis to time delay systems.

Alessandro D'Innocenzo received the Laurea degree (cum laude) in electrical engineering, in 2000, and the Ph.D. degree in electrical and information engineering, 2007, from the University of L'Aquila, Italy, where he is currently serving as an Associate Professor. His research focuses on networked systems and learning methods for control, with application to industrial and building automation.